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PREFACE

In the past few years there has been an astonishing upsurge of interest in the science dealing with the interaction of deformable conducting materials (particularly conducting fluids) and the electromagnetic field. This science has been variously called plasma dynamics, magnetohydrodynamics, hydromagnetics, magnetoaerodynamics, and Hg-dynamics. All fall short of expressing its proper scope. But however called, the science has an assured fascination for those now working in it and the promise of an exciting future for those who will enter it in years to come.

Almost certainly, the secret of success in controlling the fusion reaction of the light elements lies in mastering the use of electromagnetic fields to control the 100 odd million degree plasmas in which such fusion will take place. When such mastery is achieved, the deuterium in the sea will provide power in quantities to satisfy our wildest dreams. Missile and space engineers see in magnetoaerodynamics the possibilities of achieving greatly increased specific thrusts of rockets which will change space vehicles from flying fuel tanks to machines with respectable payloads. They also see the possibility of magnetic shields and magnetic wings for re-entry into the earth's atmosphere. More and more, astrophysicists are recognizing that magnetic fields are a powerful agent in the dynamics of our universe. Our own earth, sun, planetary system, stars, and the material between them are all profoundly influenced by magnetic fields. As engineers develop new magnetrons, klystrons, and traveling wave tubes, they too find that they must master the difficult art of understanding the interaction of beams of charged particles with electromagnetic fields.

Even now, our goals for producing high temperatures, high frequencies, high forces, and high velocities by electromagnetic fields exceed by many powers of ten those limits which we accepted complacently only a decade or so ago. Our everyday chemistry, thermodynamics, strength of materials, and so on deal with individual particle energies of the order of a few electron volts or less. To the man in the street, this level determines the nature of the physical world around him. Yet, now that we are achieving control of nuclear interactions, the level of interaction is being stepped up by a factor of over a million, and a whole new kind of daily reality lies in store for us. The intermediary for the control of this vastly increased level of activity will almost certainly be the electromagnetic field. In the absence of control, this vast difference in interaction levels manifests itself in the explosive violence of the nuclear bomb. But as we learn to bridge the gap in gentler fashion, such mastery will surely have a profound effect on the concepts we now take for granted at the lower end of the scale.

In the Fall of 1957, the Air Force Office of Scientific Research, recognizing the mounting scientific interest in the whole field of plasma dynamics, asked the National Academy of Sciences to organize an international symposium which would bring together top-ranking scientists working on the many different aspects of the subject. The National Academy appointed a committee consisting of Professor William Allis, Professor Johannes Burgers, Professor Walter Elsasser, Doctor John Pierce, and Professor Francis Clauser, chairman, to organize such a symposium. The committee decided to invite men from the fields of astrophysics, fluid mechanics, thermonuclear physics, gaseous discharges, electron beam dynamics, statistical mechanics, and aerodynamics. The list of participants is given on the following page.

The symposium was held at Woods Hole, Massachusetts during the week of June 9 to June 13, 1958. Sessions were held during mornings and evenings, with afternoons free for recreation and discussion. Only a single session took place at any given time and everyone participated in that session. No formal papers were presented. Each session had a chairman, an introductory speaker to present the subject for discussion in a provocative manner, and an editor to put the material that was presented during the discussion in suitable form for inclusion in the present volume. A transcript of the discussion was taken for the benefit of the editors. After the first day or so, the participants were on a first-name basis and the discussion became both animated and penetrating. The consensus seemed to be that it was a healthy experience to be forced to defend those concepts which you and your colleagues take for granted before a group having widely different concepts to which they will cling with a tenacity equaling your own. It is probably not too presumptuous to say that the resulting record of the symposium which is contained in this volume gives as broad a picture of the field of plasma dynamics as has yet been presented.

On behalf of the members of the organizing committee as well as the participants in the symposium, I should like to extend our grateful thanks to Mr. John Coleman and the National Academy of Sciences and to Colonel Pharo Gagge, Dr. William Otting, and the Air Force Office of Scientific Research for the great effort they put into making the symposium profitable and in providing the support which made it a success. I should also like to thank Mr. Joseph Byrne of the Addison-Wesley Publishing Company, who was present during the symposium and who played the role of executive editor in arranging the many details connected with publishing the present volume.

FRANCIS H. CLAUSER
Chairman, Symposium Committee
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Baltimore
April 22, 1959

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CONTENTS

of magnitude less than ω_p, and this may be of considerable use in studying high-density plasmas. The phase characteristic of Fig. 4–23 for no static magnetic field has the asymptotic low-frequency characteristic

$$\omega = \beta \cdot \sqrt{\frac{\log b/a}{2K_e}}\,\omega_p a, \qquad \omega \ll \omega_p, \qquad \omega/\beta \ll c, \qquad (4\text{–}36)$$

where a is the plasma radius, b the radius of the surrounding metallic cylinder, and K_e is the relative dielectric constant of the medium between a and b. An analysis of a nonuniform plasma in which the density variation is given by Eq. (4–34) shows that the low-frequency asymptotic phase velocity is proportional to the *average* plasma frequency [81]. Hence when (4–36) is applied to calculate electron density, it yields average density.

CHAPTER 5

STATISTICAL PLASMA MECHANICS

The Bridge between Particle Mechanics and Continuum Mechanics

JOHANNES M. BURGERS, *Editor*

5–1 Introduction. Many speakers at this Symposium have made use of the continuum equations to describe the flow of an ionized gas or of its constituents from the hydrodynamic point of view. Sometimes the equations have been applied in a rather detailed form; in other cases simplified forms have been preferred. The importance of the continuum equations is fully recognized and they serve as a useful tool in many problems. However, the information they offer is not always sufficient, and we must then have recourse to distribution functions for the particle velocities and introduce the Boltzmann equation.

It will be appropriate therefore to present a summary of the continuum equations and to review a number of their applications. This will bring us to several cases where the particle point of view must be called upon to help. It will also lead to a discussion of some basic concepts implied in the Boltzmann equation. It is hoped that the following account of the material considered in the session devoted to this subject will provide a background for comparison of questions treated by speakers in other sessions. In arranging the subject matter the author has taken the liberty of deviating from the order in which points of view were brought forward in the discussion, and also has interpolated a few references to existing literature where this appeared to be helpful. He also records with gratitude the help received from the speakers who supplied additional information concerning their contributions after the first draft of this chapter had been completed.

Since the continuum equations have been treated by many authors, it is not necessary to go deeply into the mathematical aspects of their deduction. The classical treatise by Chapman and Cowling, *The Mathematical Theory of Non-Uniform Gases* [1], is well known. The participants in the Symposium were very fortunate to have Professor Chapman in their midst. A different method of treatment is that in which one works with the moments of the Boltzmann equation. This is the method that was used by Maxwell. Grad, in a paper "On the Kinetic Theory of Rarefied Gases" [2],

1. S. CHAPMAN and T. G. COWLING, *The Mathematical Theory of Non-Uniform Gases*, Cambridge University Press, 1939, with notes added in 1951.

2. H. GRAD, On the kinetic theory of rarefied gases, *Comm. on Pure and Applied Math.* **2**, 331–407 (1949).

has developed this method very fully, in such a way that it can be applied to obtain a third-order approximation. The specific problems connected with this approximation, however, were not considered at the Symposium, nor were the points of view investigated in great detail by Ikenberry and Truesdell in their paper "On the Pressures and the Flux of Energy in a Gas according to Maxwell's Kinetic Theory," I and II [3].

The present author has treated the deduction of the continuum equations by means of the method of moments in a series of lectures given at the Institute for Fluid Dynamics and Applied Mathematics of the University of Maryland, in which various features connected with the application of the equations have been analyzed [4]. The deduction is based upon the approximation of the distribution function by means of a convenient expression containing (for each species of molecules) thirteen adjustable coefficients, squares and products of which are neglected. The main equations of this report will be briefly recapitulated in Sections 5–2 through 5–6; for details of the deductions the reader is referred to the lecture notes themselves.

Reference must be made to a paper by Kolodner, "Moment Description of Gas Mixtures" [5], which in a number of respects runs parallel to the treatment given in the lecture notes.

A recent paper reviewing what can be deduced more directly from Boltzmann's equation was published by Gross, "Dynamics of Electron Beams and Plasmas" [6].

3. E. Ikenberry and C. Truesdell, On the pressures and the flux of energy in a gas according to Maxwell's kinetic theory I, *J. Rational Mechanics and Analysis* **5**, 1–54 (1956); C. Truesdell, *idem* II, *ibid.*, 55–128.

4. J. M. Burgers, The application of transfer equations to the calculation of diffusion, heat conduction, viscosity and electric conductivity, parts I and II, Inst. for Fluid Dynamics and Applied Mathematics, U. of Maryland, Technical Note BN-124a/b (May 1958).

5. I. I. Kolodner, Moment description of gas mixtures I, Inst. of Math. Sci., NYU, NYO-7980 (September 1957). Grad mentioned in a letter that Kolodner in his thesis (NYU, 1950), which unfortunately was never published in full, derived the complete set of moment equations for an arbitrary mixture of gases, and calculated the relevant collision integrals for general force laws. The report of 1957 extends the results to include magnetic fields, and specializes the collision integrals to coulomb forces. A complete set of collision integrals has been evaluated, including matrix heat conductivity and viscosity coefficients for the individual gases, as well as all the conventional coefficients, such as Spitzer's and Landshoff's parallel and perpendicular resistivity. The temperature differences between the gases can be taken arbitrarily large in this approximation method.

6. E. P. Gross, Dynamics of electron beams and plasmas, *Electronic Wave Guides*, Symposium Series Vol. 8, Microwave Research Institute, Polytechnic Institute of Brooklyn, 1958.

5–2 The Boltzmann equation. Plasma dynamics, as a part of kinetic gas theory, starts from Boltzmann's equation for the behavior of the functions F_s describing the velocity distributions of the various types of particles present in a gas mixture. This equation has the form

$$\frac{\partial F_s}{\partial t} + \xi_{sh}\frac{\partial F_s}{\partial x_h} + \frac{f_{sh}}{m_s}\frac{\partial F_s}{\partial \xi_{sh}} = \left(\frac{dF_s}{dt}\right)_{\text{coll}}. \tag{5–1}$$

Subscripts s, t, ... refer to the different kinds of particles present in the gas; the ξ_{sh} are the velocity components of a particle of type s, having a mass m_s and possibly an electric charge e_s (electrostatic units); and F_s is the distribution function for the ξ_{sh}. The f_{sh} are the components of the force acting upon a particle of type s; in calculating the f_{sh}, attention is given only to so-called field forces (electric and magnetic fields, and in some cases gravity).

The forces coming into play in collisions between particles are not included in f_{sh}, but are treated separately. They are accounted for in the expression $(dF_s/dt)_{\text{coll}}$ appearing on the right side of the equation. Various forms are given for this right-hand member. Since it refers to the interaction between at least two particles (and sometimes between more than two), a proper treatment necessitates the introduction of distribution functions referring to the simultaneous position and state of motion of more than one particle. We shall come to this subject in Sections 5–18 and 5–19. For the present we follow the customary method of treatment, introducing only functions F_s for single particles.

There are cases where collisions are of such slight importance in comparison with the effects of electric and magnetic field forces that the right side can be replaced by zero. In other cases collisions may be of some importance, but it is sufficient to take account of them in an approximate way. According to a method proposed by Krook [7], this can be done by introducing a mean collision frequency and assuming that each collision redistributes the velocity components in a random manner. A refinement of this method is possible, in which mean collision frequencies are different for different combinations of collision partners, and in which the random velocity distributions resulting from the collisions are based on mean flow velocities and mean temperatures, which also can be different for different combinations of collision partners. It is even possible to introduce mean collision frequencies which are dependent upon the velocity of the particles s to which the distribution function F_s refers, the averaging being done only with respect to the various velocities of the collision partner.

7. M. Krook, Dynamics of rarefied gases, *Phys. Rev.* **99**, 1896–1897 (1955); E. P. Gross and M. Krook, Model for collision processes in gases: small-amplitude oscillations of charged two-component systems, *Phys. Rev.* **102**, 593–604 (1956).

In those cases where collisions are of great importance the standard expression introduced on the right side of Eq. (5–1) is that for *binary collisions*, in which account is taken of the geometry of the collision and of the nature of the forces acting between the colliding particles (considered as point centers of force without rotation or internal vibration):

$$\left(\frac{dF_s}{dt}\right)_{\text{coll}} = \sum_t \iint d\xi_t (F'_s F'_t - F_s F_t)\, g\, b\, db\, d\epsilon. \qquad (5\text{–}2)$$

The summation with respect to the subscript t is extended over all possible collision partners, including the case $t = s$. F'_s and F'_t are distribution functions having as arguments the velocities of the particles s and t, respectively, after the collision. The relative velocity $\mathbf{g}$ is defined by

$$g_h = \xi_{th} - \xi_{sh}.$$

The parameters b and ϵ specify the geometry of the collision, b being the distance of the particle s from the original line of motion of the particle t (before the collision started) and ϵ representing an angle of position. The element $b\, db\, d\epsilon$ [to which the second integral sign on the right side of Eq. (5–2) refers, the first integral referring to $d\xi_t$] is an element of the collisional cross section. The velocity components ξ'_{sh}, ξ'_{th} of the particles after the collision are influenced by the relative velocity $\mathbf{g}$, by the magnitude of b, and by the force field coming into play in the collision process.

In calculating the velocities after the collision, attention usually is given only to conservative collisions, in which the particles suffer no intrinsic change. There can be, however, collisions in which energy is exchanged with interior degrees of freedom, in such a way that the rotational, vibrational, or electronic state of one or both of the partners is altered; or the collision can lead to dissociation, ionization, recombination, a chemical reaction, or exchange of energy with a radiation field. At sufficiently high temperatures, where quantum effects can be neglected, it is possible to take account of exchange of energy with rotational motion and with vibrations in an approximate but on the whole satisfactory way. When change of electronic state or a chemical reaction can occur, a detailed treatment becomes very difficult. Quantum mechanics is involved, and the presence of another particle besides the direct collision partner may be of importance. So far the only practical way of treating phenomena involving such reactions within the compass of plasma dynamics would be to introduce mean collision frequencies (or, alternatively, mean collision cross sections) for each phenomenon to be considered, and to assume that the particles resulting from the reaction will have a random distribution of their velocity components around some mean flow velocity

and corresponding to some mean temperature, depending upon the type of reaction investigated. Here these problems will be left aside.

There are cases where the assumption of discrete binary collisions appears to be inappropriate, and where the almost simultaneous effect of a great number of weak collisions must be considered. The form then given to the right side of Eq. (5–1) is the Fokker-Planck expression for the average resultant effect of a large number of small interactions, which will be considered in Sections 5–18, 5–20, 5–21, and 5–22.

5–3 Equations of transfer (moments of the Boltzmann equation). The equations of motion for the gas are deduced by taking the moments of both sides of the Boltzmann equation, with reference to factors which are functions of the velocity components. The equations obtained in this way are called "equations of transfer," since the integral of the right side gives information concerning the transfer of momentum, energy, or some other quantity in the collisions.

The left side of the Boltzmann equation is linear in the distribution function F_s and its moments can be obtained by a straightforward integration. It should be observed that when the multiplying factor is of degree n in the velocity components, there always appear spatial derivatives of moments of degree $n + 1$, in consequence of the presence of the factor ξ_{sh} before $\partial F_s/\partial x_h$ in Eq. (5–1).

The treatment of the right side of Boltzmann's equation brings greater problems. Apart from the trivial case in which the right side is absent, the integration is simple only in those cases where the right side can be expressed with the aid of Krook's approximate formulas. It is also relatively simple when a Fokker-Planck expression is used on the right side, since this expression again is linear in the distribution function.

With the form (5–2) of the right side, which is the one that has received most attention, it is in general not possible to evaluate the necessary integrals, unless series developments are introduced for the distribution functions F_s, F_t, F'_s, F'_t. The only exception is the special case of molecules repelling each other with a force inversely proportional to the fifth power of the distance (so-called Maxwellian molecules). The introduction of series leads to problems concerning the most appropriate method of solution of the equations and concerning the convergence of the solutions so constructed. The circumstance that the left side of the integrated equations always contains certain moments of a degree higher than those appearing in the main terms complicates the problem of convergence. These questions are considered in references [1], [2], [3], and in other papers. Fortunately, in plasma dynamics, the problems of higher approximations and of convergence thus far have not forced their way into the analysis. In the following sections they will be left aside.

5–4 Expansion of the distribution function. In the "first-order" treatment of the right side of the Boltzmann equation, the distribution function is assumed to be of the form

$$F_s = F_{s0}(1 + \phi_s), \tag{5–3}$$

where ϕ_s is a small quantity whose squares and products can be neglected. The functions F_{s0}, ϕ_s are given below in Eqs. (5–13) and (5–14).

We write N_s for the number density per unit volume of the particles of type s. The mass density is given by $\rho_s = N_s m_s$ and the charge density by $\rho_{es} = N_s e_s$ for each type separately. For the gas as a whole we have

$$N = \sum_s N_s; \qquad \rho = \sum_s \rho_s; \qquad \rho_e = \sum_s \rho_{es}.$$

For each species of particles we define mean flow velocity components u_{sh} as the mean value of the particle velocities:

$$u_{sh} = \overline{\xi_{sh}}. \tag{5–4}$$

A mean flow velocity for the gas as a whole is defined by

$$\rho u_h = \sum_s \rho_s u_{sh}. \tag{5–5}$$

We then write

$$w_{sh} = u_{sh} - u_h, \tag{5–6}$$

and call the w_{sh} the components of the *diffusion velocity* of the particles of type s with respect to the mean mass flow. We also write

$$c_{sh} = \xi_{sh} - u_h, \qquad \text{with} \qquad \overline{c_{sh}} = w_{sh}. \tag{5–7}$$

It is convenient to introduce the following quantities referring to the mean mass flow:

$$\epsilon = \frac{\partial u_h}{\partial x_h}; \qquad \epsilon_{hk} = \frac{\partial u_k}{\partial x_h} + \frac{\partial u_h}{\partial x_k} - \frac{2}{3}\delta_{hk}\epsilon; \tag{5–8}$$

and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_h \frac{\partial}{\partial x_h}.$$

The electric current strength is given by

$$J_{sh} = \rho_{es} w_{sh}, \qquad \text{with} \qquad J_h = \sum_s J_{sh}. \tag{5–9}$$

The temperature T_s of the particles of type s (with reference to the mass flow velocity) is defined by

$$\tfrac{3}{2}\kappa T_s = \tfrac{1}{2} m_s \overline{c_s^2}, \tag{5–10}$$

where κ is Boltzmann's constant. The mean temperature of the gas as a whole follows from

$$NT = \sum_s N_s T_s. \tag{5–11}$$

It is convenient to write

$$a_s^2 = \frac{2\kappa T}{m_s}. \tag{5–12}$$

Here a_s is defined with the aid of the mean temperature T of the gas (a different definition will be used temporarily in Section 5–15).

For F_{s0} we now assume

$$F_{s0} = \frac{N_s}{\pi^{3/2} a_s^3} \exp\left(\frac{-c_s^2}{a_s^2}\right). \tag{5–13}$$

Evidently F_{s0} can be expressed as a function of the ξ_{sh}; it then has as parameters the number density N_s, the mean mass flow velocity u_h, and the mean temperature T.

The following expression is used for the function ϕ_s:

$$\phi_s = A_{sh} c_{sh} + B_s(c_s^2 - \tfrac{3}{2}a_s^2) + B_{shk}(c_{sh}c_{sk} - \tfrac{1}{3}\delta_{hk}c_s^2) + \\ + C_{sh}(c_s^2 c_{sh} - \tfrac{5}{2}a_s^2 c_{sh}). \tag{5–14}$$

The coefficients A_{sh}, etc., are functions of the coordinates and the time, to be found from the equations which will be developed. In working out the integral of the right side of the Boltzmann equation, products and squares of these coefficients are neglected. When the development is limited to the terms occurring in (5–14), it is not necessary to introduce Hermite or Sonine polynomials. The degree of approximation reached in the results is the same as that given by Chapman and Cowling in the treatment of a mixture of gases as presented in Chapter 9 of reference [1].

By calculating the mean values of the c_{sh} and of powers and products of these quantities on the basis of the distribution function (5–3) in conjunction with (5–12) through (5–14) it is possible to express the coefficients A_{sh}, etc., with the aid of the diffusion velocities w_{sh}, the temperature differences $T_s - T$, the deviation components of the pressure tensor $(P_s)_{hk}$ [defined in equations (5–16) below], and the components of the heat flow, q_{sh}. This makes it possible to bring the resulting equations into a form directly related to physical quantities.

The following multiplicative factors will be used:

$$m_s; \qquad m_s c_{sh}; \qquad \tfrac{1}{2} m_s c_s^2; \qquad m_s c_{sh} c_{sk}; \qquad \tfrac{1}{2} m_s c_s^2 c_{sh}.$$

In all, thirteen equations are obtained for each constituent of the gas mixture.

5–5 Additional definitions. Mean values indicated by a bar over a quantity refer to averaging over all velocity components of the particles of the type considered, making use of the distribution function F_s for these particles, as defined by (5–3) in combination with (5–12) through (5–14).

Quantities characterizing field forces are:

Acceleration of gravity: G_h

Electric field strength: E_h (electrostatic units)

Magnetic flux: B_h (magnetic units; the magnetic permeability of space is taken equal to unity and effects of magnetic polarization are left out of account).

We write

$$E_h^* = E_h + \frac{(\mathbf{u} \times \mathbf{B})_h}{c}. \tag{5–15}$$

Components of the pressure tensor:

$$\begin{gathered} (p_s)_{hk} = N_s m_s \overline{c_{sh} c_{sk}}, \\ p_s = \tfrac{1}{3}(p_s)_{hh} = \tfrac{1}{3} N_s m_s \overline{c_s^2} = N_s \kappa T_s, \\ (P_s)_{hk} = (p_s)_{hk} - \delta_{hk} p_s, \\ p_{hk} = \sum_s (p_s)_{hk}, \qquad P_{hk} = \sum_s (P_s)_{hk}, \qquad p = \sum_s p_s = N \kappa T. \end{gathered} \tag{5–16}$$

Moments of the third degree and heat flow components:

$$\begin{gathered} (p_s)_{hkl} = N_s m_s \overline{c_{sh} c_{sk} c_{sl}}, \\ q_{sl} = \tfrac{1}{2}(p_s)_{hhl} = \tfrac{1}{2} N_s m_s \overline{c_s^2 c_{sl}}. \end{gathered} \tag{5–17}$$

Residual heat flow vector:

$$q_{sh}^* = q_{sh} - \tfrac{5}{2} p_s w_{sh} = q_{sh} - \tfrac{5}{2} N_s \kappa T_s w_{sh}; \tag{5–18a}$$

reduced residual heat flow vector:

$$r_{sh} = \frac{q_{sh}^*}{N_s \kappa T} = \frac{\tfrac{1}{2} m_s \overline{c_s^2 c_{sh}}}{\kappa T} - \frac{5}{2} w_{sh} \tag{5–18b}$$

[with neglect of a second order term $\frac{5}{2}(T_s/T - 1) w_{sh}$].

For the treatment of binary collisions between particles s and t (where t may be the same as s) we need

$$m_0 = m_s + m_t, \qquad x = m_t/m_0, \qquad y = m_s/m_0, \qquad \mu = m_s m_t / m_0,$$

$$a_s^2 = 2\kappa T/m_s \quad \text{[see Eq. (5–12)]}, \qquad \alpha^2 = 2\kappa T/\mu.$$

Collision cross section for a given relative velocity g:

$$S_{st}^{(l)} = S_{ts}^{(l)} = 2\pi \int (1 - \cos^l \chi) b \, db; \tag{5–19a}$$

average collision cross section for all possible relative velocities:

$$Z_{st}^{(lj)} = Z_{ts}^{(lj)} = \frac{4}{\pi^{1/2} \alpha^{2j+4}} \int_0^\infty dg \, g^{2j+3} \exp\left(\frac{-g^2}{\alpha^2}\right) S_{st}^{(l)}. \tag{5–19b}$$

For *coulomb interactions between charged particles* (both for attraction and repulsion) we introduce the Debye shielding distance:

$$r_D = \left(\frac{\kappa T}{4\pi \sum_s N_s e_s^2}\right)^{1/2}. \tag{5–20}$$

We also write

$$\Lambda = \frac{3\kappa T}{e^2} r_D. \tag{5–21}$$

Then [8]:

$$S_{st}^{(1)} = 4\pi \left(\frac{e_s e_t}{\mu g^2}\right)^2 \ln \Lambda; \qquad S_{st}^{(2)} = 8\pi \left(\frac{e_s e_t}{\mu g^2}\right)^2 \ln \Lambda; \tag{5–22a}$$

$$Z_{st}^{(11)} = Z_{st}^{(12)} = Z_{st}; \qquad Z_{st}^{(13)} = Z_{st}^{(22)} = 2 Z_{st}, \tag{5–22b}$$

with

$$Z_{st} = 2\sqrt{\pi} \left(\frac{e_s e_t}{\kappa T}\right)^2 \ln \Lambda. \tag{5–22c}$$

8. More refined expressions are given by CHAPMAN and COWLING (Ref. 1, p. 179; the notation is different from that used in the text). Their expressions make the ratio $S^{(2)}/S^{(1)}$ slightly less than 2. But even these expressions are based upon assumptions concerning the best way of cutting off divergent integrals.

It should be kept in mind that these expressions do not hold for charged particles still carrying an electronic system (e.g., singly charged ions of an element other than H); in that case the electronic system will exert forces of another type when the particles approach each other closely and the cross sections S and Z have additional terms which may not vanish for infinite relative speed or infinite temperature (compare Section 5–15, last paragraph).

Derived quantities (all symmetric in s and t):

$$z_{st} = 1 - \frac{2}{5}\frac{Z_{st}^{(12)}}{Z_{st}^{(11)}}; \qquad z'_{st} = 1 - \frac{4}{35}\frac{Z_{st}^{(13)}}{Z_{st}^{(11)}};$$
$$z''_{st} = \frac{Z_{st}^{(22)}}{Z_{st}^{(11)}}; \qquad \zeta_{st} = 1 + 5z_{st} - \frac{7}{2}z'_{st}; \tag{5–23}$$

$$Y_s = 3y + \zeta_{st}(m_t^2/m_s m_0); \qquad Y_t = 3x + \zeta_{st}(m_s^2/m_t m_0);$$
$$y_s = (3 + \zeta_{st})x; \qquad y_t = (3 + \zeta_{st})y. \tag{5–24}$$

Coefficient of friction for the diffusion of particles s and t relative to each other:

$$K_{st} = K_{ts} = \tfrac{2}{3}\mu\alpha N_s N_t Z_{st}^{(11)}. \tag{5–25}$$

Components of the tensor $(M_s)_{hk}$ occurring in (5–34) below:

h, k	
11	$(2e_s/m_s c)[(P_s)_{12}B_3 - (P_s)_{13}B_2]$
22	$(2e_s/m_s c)[(P_s)_{23}B_1 - (P_s)_{12}B_3]$
33	$(2e_s/m_s c)[(P_s)_{13}B_2 - (P_s)_{23}B_1]$
12, 21	$(e_s/m_s c)[(P_s)_{13}B_1 - (P_s)_{23}B_2 - (P_s)_{11}B_3 + (P_s)_{22}B_3]$
13, 31	$(e_s/m_s c)[-(P_s)_{12}B_1 + (P_s)_{11}B_2 - (P_s)_{33}B_2 + (P_s)_{23}B_3]$
23, 32	$(e_s/m_s c)[-(P_s)_{22}B_1 + (P_s)_{33}B_1 + (P_s)_{12}B_2 - (P_s)_{13}B_3]$

(5–26)

The off-diagonal terms are symmetrical in h and k, and the trace of the tensor is zero. In the place of the $(P_s)_{hk}$ we might just as well have written the $(p_s)_{hk}$.

5–6 Resulting equations. (A) *Equations for the flow of the gas as a whole.*

Continuity equation:

$$\frac{D\rho}{Dt} + \rho\epsilon = 0. \tag{5–27}$$

Momentum equation:

$$\rho\frac{Du_h}{Dt} + \frac{\partial p_{ih}}{\partial x_i} - \rho G_h - \rho_e E_h^* - \frac{(\mathbf{J}\times\mathbf{B})_h}{c} = 0. \tag{5–28}$$

Equation for the internal energy:

$$\frac{D}{Dt}\left(\frac{3}{2}p\right) + \frac{5}{2}p\epsilon + \frac{1}{2}P_{ij}\epsilon_{ij} + \frac{\partial q_i}{\partial x_i} - E_i^* J_i = 0. \tag{5–29}$$

An alternative is the equation for the total energy:

$$\rho\frac{D}{Dt}\left(\frac{3p}{2\rho} + \frac{1}{2}u^2\right) + \frac{\partial}{\partial x_i}(u_j p_{ij}) + \frac{\partial q_i}{\partial x_i} - \rho G_i u_i - E_i(J_i + \rho_e u_i) = 0. \tag{5–30}$$

(B) *Equations for the separate constituents of the gas mixture.*

Continuity equations:

$$\frac{D\rho_s}{Dt} + \rho_s\epsilon + \frac{\partial}{\partial x_i}(\rho_s w_{si}) = 0. \tag{5–31}$$

Momentum equations for the diffusion velocities, after elimination of Du_h/Dt with the aid of (5–28):

$$\left[\frac{D}{Dt}(\rho_s w_{sh}) + \rho_s w_{sh}\epsilon + \rho_s w_{si}\frac{\partial u_h}{\partial x_i} + \frac{\partial (P_s)_{ih}}{\partial x_i} - \frac{\rho_s}{\rho}\frac{\partial P_{ih}}{\partial x_i}\right]$$
$$+ \frac{\partial p_s}{\partial x_h} - \frac{\rho_s}{\rho}\frac{\partial p}{\partial x_h} - \left(\rho_{es} - \frac{\rho_s\rho_e}{\rho}\right)E_h^* - \frac{1}{c}\left\{\left(\mathbf{J}_s - \frac{\rho_s}{\rho}\mathbf{J}\right)\times\mathbf{B}\right\}_h =$$
$$= -\sum_t K_{st}\{(w_{sh} - w_{th}) - z_{st}(xr_{sh} - yr_{th})\}. \tag{5–32}$$

Equation for the internal energy:

$$\frac{1}{2}(P_s)_{ij}\epsilon_{ij} + \frac{\partial q_{si}}{\partial x_i} + \rho_s w_{si}\frac{Du_i}{Dt} + \frac{D}{Dt}\left(\frac{3}{2}p_s\right) + \frac{5}{2}p_s\epsilon - \rho_s G_i w_{si} - E_i^* J_{si} =$$
$$= -3\sum_t K_{st}\frac{\kappa(T_s - T_t)}{m_0}. \tag{5–33}$$

Equation for the viscous stresses:

$$\left[\frac{D}{Dt}(P_s)_{hk} + (P_s)_{hk}\epsilon + \frac{\partial}{\partial x_l}\left\{(p_s)_{hkl} - \frac{2}{3}\delta_{hk}q_{sl}\right\} + \right.$$

$$+ \rho_s\left(w_{sk}\frac{Du_h}{Dt} + w_{sh}\frac{Du_k}{Dt} - \frac{2}{3}\delta_{hk}w_{sl}\frac{Du_l}{Dt}\right) +$$

$$\left. + (P_s)_{hl}\frac{\partial u_k}{\partial x_l} + (P_s)_{kl}\frac{\partial u_h}{\partial x_l} - \frac{2}{3}\delta_{hk}(P_s)_{lj}\frac{\partial u_j}{\partial x_l}\right] +$$

$$+ p_s\epsilon_{hk} - \rho_s\left(G_hw_{sk} + G_kw_{sh} - \frac{2}{3}\delta_{hk}G_iw_{si}\right) -$$

$$- \left(E_h^*J_{sk} + E_k^*J_{sh} - \frac{2}{3}\delta_{hk}E_i^*J_{si}\right) - (M_s)_{hk} =$$

$$= -\frac{\sqrt{2}}{5}N_s\alpha_sZ_{ss}^{(22)}(P_s)_{hk} -$$

$$- \sum_t \frac{2}{5}N_sN_t\,\alpha Z_{st}^{(22)}\{x^2(P_s)_{hk}/N_s + xy(P_t)_{hk}/N_t\} -$$

$$- \sum_t \frac{4}{3}N_sN_t\,\frac{m_sm_t}{m_0^2}\,\alpha Z_{st}^{(11)}\{(P_s)_{hk}/N_s - (P_t)_{hk}/N_t\}. \tag{5-34}$$

Equations for the components of the heat flow:

$$\left[\frac{D}{Dt}(q_{sh}) + q_{sh}\epsilon + q_{sl}\frac{\partial u_h}{\partial x_l} + (p_s)_{hlj}\frac{\partial u_j}{\partial x_l} + \frac{1}{2}\frac{\partial}{\partial x_l}(P_s)_{hlkk}\right.$$

$$\left. - \frac{5}{2}\frac{p_s}{\rho_s}\frac{\partial}{\partial x_l}(P_s)_{hl} + (P_s)_{hl}\frac{Du_l}{Dt} - \frac{5}{2}p_s\left\{\frac{1}{\rho_s}\frac{D}{Dt}(\rho_sw_{sh}) + w_{sh}\epsilon + w_{sl}\frac{\partial u_h}{\partial x_l}\right\}\right] +$$

$$+ \frac{5}{2}\frac{N_s\kappa^2T_s}{m_s}\frac{\partial T_s}{\partial x_h} - G_i(P_s)_{hi} - \frac{\rho_{es}}{\rho_s}E_i^*(P_s)_{hi} - \frac{\rho_{es}}{\rho_s}\frac{(\mathbf{q}_s^* \times \mathbf{B})_h}{c} =$$

$$= \sum_t K_{st}z_{st}\frac{5\kappa Tm_t}{2m_0m_s}(w_{sh} - w_{th}) - \frac{2}{5}K_{ss}\frac{\kappa T}{m_s}z_{ss}''r_{sh} -$$

$$- \sum_t K_{st}\frac{\kappa T}{m_0}\left\{\frac{4}{5}xz_{st}''(r_{sh} + r_{th}) + Y_sr_{sh} - y_sr_{th}\right\}. \tag{5-35}$$

In the summations with respect to t in Eqs. (5–34) and (5–35) the case $t = s$ must be excluded, since it has been introduced as a separate term.

The left sides of Eqs. (5–27) through (5–35) are independent of the assumptions introduced to describe the effect of the collisions. The right sides of Eqs. (5–32) through (5–35) have been based, as stated before, upon Eq. (5–2), together with (5–3), (5–12), (5–13), and (5–14). The right sides of Eqs. (5–27) through (5–31) are zero, independently of any assumption about the form of the right side of the Boltzmann equation.

No terms have been omitted in the equations, but in many cases various terms on the left sides can be left out of account. In order to arrive at the usual equations for the calculation of diffusion coefficients (including the coefficient of electric conductivity), viscosity, and heat flow, the terms between [ ] should be omitted. In certain cases, however, it may be interesting to retain the time derivatives in order to obtain information concerning relaxation times. In the energy equation (5–33) one will usually omit the third term of the first line, but the term $\partial q_{si}/\partial x_i$ must be retained when the divergence of the heat flow is of importance, and the term $\frac{1}{2}(P_s)_{ij}\epsilon_{ij}$ when viscous dissipation has to be taken into consideration. These terms have been retained also in Eq. (5–29) for the gas as a whole [the sum $\sum_s\rho_sw_{si}(Du_i/Dt) = 0$ in consequence of (5–5) and (5–6)].

5–7 Alternative form of the equations of motion. In cases where collisions between particles are almost or wholly negligible, the diffusion velocities w_{sh} may become of the same order of magnitude as the mean mass flow velocity u_h. In such cases it may be less convenient to use the equations of motion in the form given in the preceding section, and one may prefer equations based upon the velocity components u_{sh} for each separate component, as defined by Eq. (5–4). When this is done, a number of quantities must be defined differently from before.

The random or peculiar velocity of the particles of type s will now have the value

$$c_{sh}^0 = \xi_{sh} - u_{sh}, \tag{5-36a}$$

so that

$$c_{sh}^0 = c_{sh} - w_{sh}, \qquad \text{with} \qquad \overline{c_{sh}^0} = 0. \tag{5-36b}$$

The components of the pressure tensor and those of the heat flow vector must be defined with reference to the flow velocities u_{sh}. We obtain

$$\begin{aligned}(p_s^0)_{hk} &= N_sm_s\overline{c_{sh}^0c_{sk}^0} = (p_s)_{hk} - \rho_sw_{sh}w_{sk},\\ p_s^0 &= \tfrac{1}{3}N_sm_s\overline{(c_s^0)^2} = p_s - \tfrac{1}{3}\rho_sw_s^2,\\ (P_s^0)_{hk} &= (p_s^0)_{hk} - \delta_{hk}p_s^0;\end{aligned} \tag{5-37}$$

$$(q_s^0)_h = \tfrac{1}{2}N_sm_s\overline{(c_s^0)^2c_{sh}^0} = q_{sh} - \tfrac{3}{2}p_sw_{sh} - (p_s)_{hi}w_{si} + \rho_sw_s^2w_{sh}. \tag{5-38}$$

It must be observed that now it is *not* possible to sum the pressure components or the components of the heat flow for the various species of particles, since they are referred to different flow velocities.

We further write

$$\frac{D_s}{Dt} = \frac{\partial}{\partial t} + u_{si}\frac{\partial}{\partial x_i}$$

(differentiation with respect to time, following the motion of a particular constituent of the mixture).

The components of the electric current carried by the particles of type s will be defined by

$$J^0_{sh} = \rho_{es}u_{sh} = \rho_{es}u_h + J_{sh}, \tag{5–39}$$

so that the new quantities J^0_{sh} combine what in the other description are distinguished as convection current and conduction current.

The equations obtain the following forms:

Equation of continuity [replacing former Eqs. (5–27) and (5–31)]:

$$\frac{\partial \rho_s}{\partial t} + \frac{\partial}{\partial x_i}(\rho_s u_{si}) = 0$$

or

$$\frac{D_s\rho_s}{Dt} + \rho_s\frac{\partial u_{si}}{\partial x_i} = 0. \tag{5–40}$$

Momentum equation [replacing former Eqs. (5–28) and (5–32)]:

$$\rho_s\frac{D_s(u_{sh})}{Dt} + \frac{\partial}{\partial x_i}\{(p^0_s)_{ih}\} - \rho_s G_h - \rho_{es}E_h - \frac{(J^0_s \times B)_h}{c} = \text{effect of collisions.} \tag{5–41}$$

Energy equation [replacing former Eqs. (5–29) and (5–33)]:

$$\frac{D_s}{Dt}\left(\frac{3}{2}p^0_s\right) + \frac{5}{2}p^0_s\frac{\partial u_{si}}{\partial x_i} + (P^0_s)_{ij}\frac{\partial u_{sj}}{\partial x_i} + \frac{\partial q^0_{si}}{\partial x_i} = \text{effect of collisions.} \tag{5–42}$$

Alternative form [replacing Eq. (5–30)]:

$$\rho_s\frac{D_s}{Dt}\left(\frac{3}{2}\frac{p^0_s}{\rho_s} + \frac{u_s^2}{2}\right) + \frac{\partial}{\partial x_i}\{u_{sh}(p^0_s)_{hi}\} + \frac{\partial q^0_{si}}{\partial x_i} - \rho_s G_i u_{si} - E^i J^0_{si} = \text{effect of collisions.} \tag{5–43}$$

It is not possible to obtain expressions for the right sides of these equations by a direct transformation of the right sides of the equations of Section 5–6, because the latter have been derived with the aid of approximations holding only when the relative diffusion velocities of the various constituents and their differences in temperature are small. An approximation suitable when the differences in flow velocity and in temperature are arbitrary will be given in Section 5–15.

5–8 Other procedures used in the deduction of transfer equations. (The observations in this section have been taken mainly from contributions to the discussion by Max Krook.) The procedure applied in Sections 5–4, 5–5, and 5–6 was based upon the assumption that the distribution functions F_s could be developed into series and that the coefficients appearing in these series are sufficiently small so that an approximation could be constructed, proceeding according to the powers of a conveniently chosen parameter. The parameter chosen is related to the inverse of the collision frequency; the equations given in Section 5–6 represent a first step in the procedure, in which only terms of the first order are retained. The Chapman-Enskog method, extensively treated in reference [1], is a device for obtaining successively higher approximations. The difficulties encountered are due to two circumstances: the nonlinear form of the right side of the Boltzmann equation, and the appearance on the left side of moments of an order one unit higher than the order of the main terms. This necessitates the introduction of specially designed cutoff procedures in order to arrive at a regular increase of precision in the results. A mathematical investigation into the convergence of this procedure, adapted to the case of Maxwellian molecules, has been developed by Truesdell in reference [3].

Other methods have been proposed to cope with the cutoff problem. In most of them, a convenient particular form is chosen for the distribution function, containing polynomials with a number of adjustable coefficients which are supposed to be functions of the coordinates and the time. This distribution function is substituted into the Boltzmann equation and the moments of both sides are calculated with full account of the terms of the second degree on the right side. It is always possible to construct as many moment equations as there are unknown coefficients in the development, and in this way a closed set of differential equations of the second degree is obtained. With modern computational machinery, such equations can be attacked. The degree of precision obtained is determined by the number of coefficients that have been introduced, and there is no longer the restriction that these coefficients should be small.

The thirteen-moment method devised by Grad in reference [2] and the method worked out by Mott-Smith [9] are based upon this idea [10].

Krook observed that both the Chapman-Enskog method and Grad's procedure aim at the deduction of flow equations that hold within the field, independently of boundary conditions. Their application to the boundary conditions relevant to a particular problem is considered at a later stage. This may not always be the most convenient approach when the situation in the field leads to the appearance of sharp gradients of velocity or temperature. It may then be more profitable to choose a representation of the distribution function in which the particular character of the boundary data of the problem has been taken into account.

As an example, consider the problem of determining the structure of a plane stationary shock wave in an ionized gas, in the absence of external fields. With the definition

$$\Phi_s(\xi, \mathbf{u}, T) = (m_s/2\pi\kappa T)^{3/2} \exp\left[-m_s(\xi - \mathbf{u})^2/2\kappa T\right],$$

the kinetic equations and the Poisson equation

$$dE/dx = 4\pi \sum_s N_s e_s$$

have to be solved subject to the boundary conditions

$$F_s(\xi, -\infty) = N_s(-\infty)\Phi_s(\xi, u_1\mathbf{i}, T_1), \qquad E(-\infty) = 0;$$

$$F_s(\xi, +\infty) = \frac{u_1}{u_2} N_s(-\infty)\Phi_s(\xi, u_2\mathbf{i}, T_2), \qquad E(+\infty) = 0.$$

Defining the moments $M^{(s)}_{m,n}$ of the distribution functions and the interaction moments $I^{(s)}_{m,n}$ by the equations

$$M^{(s)}_{m,n} = \int \xi_1^m \xi^{2n} F_s \, d\xi,$$

$$I^{(s)}_{m,n} = \int \xi_1^m \xi^{2n} (dF_s/dt)_{\text{coll}} \, d\xi,$$

9. H. M. Mott-Smith, The solution of the Boltzmann equation for a shock wave, *Phys. Rev.* **82**, 885–892 (1951).

10. Grad made the point that the Hilbert-Chapman-Enskog method of solution gives an account of only a very special subclass of solutions of the Boltzmann equation. He has considered this subject in great detail in his article "Principles of the kinetic theory of gases," *Handbuch der Physik*, **XII** (1958). It is therefore that the procedure of using a polynomial expansion of the general solution of the Boltzmann equation, rather than a "normal" solution, is of so much importance.

we can derive, from the Boltzmann equation, the following sequence of moment equations:

$$\frac{dM^{(s)}_{m+1,n}}{dx} - \frac{e_s}{m_s} E[mM^{(s)}_{m-1,n} + 2nM^{(s)}_{m+1,n-1}] = I^{(s)}_{m,n}$$

$$\text{with } m, n = 0, 1, 2, \ldots; \qquad s = 1, 2. \tag{5–44a}$$

To construct a determinate set of N equations that involve only N moments, we set

$$F_s(\xi, x) = \sum_{\lambda=1}^{2} \Phi_s(\xi, u_\lambda\mathbf{i}, T_\lambda) \sum_{m,n} A^{(s,\lambda)}_{m,n}(x) \xi_1^m \xi^{2n}, \tag{5–44b}$$

where there are altogether N parameters $A^{(s,\lambda)}_{m,n}$ which are interpreted to be arbitrary, unspecified functions of x only. We choose N particular moments $M^{(s)}_{p,q}$ to serve as macroscopic state variables, along with E. The N parameters can then be expressed as linear homogeneous functions of the state variables. Thus any "extraneous moments" can be eliminated from a basic set of N of the moment equations (5–44a). Moreover, the interaction moments in these equations can be evaluated explicitly as quadratic functions of the parameters $A^{(s,\lambda)}_{m,n}$, and hence as quadratic functions of the state variables. In this way, for any selected order of approximation N, the problem reduces to the solution of a determinate set of $N + 1$ ordinary differential equations—the Poisson equation and N moment equations. The four conservation requirements permit the elimination of four of the state variables. Boundary conditions for the moments can be obtained from the formulas given above.

The approximating form (5–44a) may be motivated in the following way. As we proceed from $x = -\infty$ to $x = +\infty$, the Maxwellian distributions $\Phi_s(\xi, u_1\mathbf{i}, T_1)$ decay nonuniformly with respect to the magnitude and the direction of the molecular velocities, while the Maxwellian distributions $\Phi_s(\xi, u_2\mathbf{i}, T_2)$ build up, also nonuniformly. The nonuniformity of the decay and the buildup are represented by the modifying polynomials in (5–44a) that multiply the two types of Maxwell functions.

The approximation procedure may be generalized by adding to the form (5–44a) other modified Maxwell functions, based on other velocity-temperature pairs.

In the problem of Couette flow, with or without heat transfer, we must in general work with the auxiliary distribution functions $F_+(\xi, x)$ defined only for $\xi_1 > 0$ and $F_-(\xi, x)$ defined only for $\xi_1 < 0$, since $F(\xi, x)$ is singular on the plane $\xi_1 = 0$ in velocity space. When the distance between the walls is a few mean free paths, the approximation is improved considerably by adding to sums of the form (5–44a) for F_+ and F_-,

modified Maxwell functions corresponding to the local flow velocity and local kinetic temperature. This type of generalization is now being applied to the shock wave problem.

Eugene Gross added some further comments and stated that his views on the problem of solving the Boltzmann equation for definite microscopic boundary conditions were in complete agreement with those of Krook, except for some differences of emphasis. These views are set down in an article on boundary value problems in the kinetic theory of gases [11]. As an example, the single relaxation time collision expression is used instead of the Boltzmann collision kernel. The most rigorous way of solving these problems is by the integral equation method described by Krook in reference [7]. The method can be carried through for several collision models, but it is probably not practicable for the rigorous Boltzmann expression. It gives an account not only of the half-range character of the distribution function near a wall, but also of the nonanalytic dependence of the distribution function on velocity. A simpler procedure is to take account of the half-range character but to slur over the correct analytic behavior with respect to velocity, hoping that the errors will not be too great. There is some justification for this if one is interested in only low-velocity moments of the distribution function, such as density, stresses, kinetic temperature, and heat flow. This justification can be inferred by examining the similar but simpler mathematical theory of radiative transfer, where both exact and approximate solutions have been studied in detail. But even within the framework of the approximate half-range methods several different approaches are possible. Krook and his coworkers are emphasizing half-range generalizations of the Mott-Smith approach to the theory of the structure of a shock front. This approach restricts the functional form of the half-range functions. The method is neat and leads to a relatively simple theory of the difficult nonlinear problems of gas dynamics. However, the validity of the approach seems hard to ascertain.

Gross has concentrated on a systematic half-range approach. The theory is considerably more cumbersome but more rigorous. Complete analyses are possible, however, for problems where the linearized Boltzmann equation applies. The full Boltzmann collision kernel can be handled. Two examples of this analysis have been published [12]. The detailed results of the first paper provided some insight into the remarkable range of validity of the slip flow approximation of continuum theory. The second paper shows that the integral equation approach is needed to discuss properly a time-dependent boundary value problem for time intervals less than a few collision times.

11. E. P. Gross, E. A. Jackson, and S. Ziering, Boundary value problems in kinetic theory of gases, *Ann. phys.* **1**, 141–167 (1957).

12. E. P. Gross and S. Ziering, Kinetic theory of linear shear flow, *Physics of Fluids* **1**, 215–224 (1958); E. P. Gross and E. A. Jackson, Kinetic theory of the impulsive motion of an infinite plate, *Physics of Fluids* **1**, 318–328 (1958); E. A. Jackson, Ph.D. Thesis, Syracuse U. (1958).

In an unfinished publication Jackson and Gross have investigated the justification of some of Krook's generalized collision models. This is done by showing that these models can be obtained by making a bi-orthogonal expansion of the exact Boltzmann collision kernel. When this expansion is approximated by a finite sum, Krook's models result. Some of the more sophisticated models are excellent approximations to the Boltzmann equation.

5–9 Problems concerning electron beams. In the discussion Gross pointed out that one of the most important aspects of the Boltzmann equation is that it can lead to a closed set of equations in two limiting conditions: one can go to the hydrodynamic limit, and one can go to the limit where individual particles are followed. It may be interesting to devote some attention to the particle aspect, in view of the subjects which have been considered in other sessions of this Symposium. When the Boltzmann equation is combined with the idea of a self-consistent electric field, we arrive at the basis for the descriptions used in electron beam theory and in a great deal of plasma physics. Herman Haus mentioned that much of the experimental work done on electron beams can be considered as a test of certain solutions of the Boltzmann equation for cases where short-range collisions can be neglected. In analyzing experimental situations, the velocity distribution is also quite often neglected, but when that is done we have put aside the Boltzmann equation itself as well. So let us keep the velocity distribution in view and reduce the Boltzmann equation to its left side only. To obviate the difficulties resulting when there is a magnetic field, let us exclude magnetic effects and write the equation in the form (for electrons only)

$$\frac{\partial F}{\partial t} + \xi_h \frac{\partial F}{\partial x_h} - \frac{eE_h}{m} \frac{\partial F}{\partial \xi_h} = 0. \tag{5–45}$$

This equation must be combined with Maxwell's equations in order to link the electric field with the charge density. For simplicity we consider the positive ions as giving a uniform background charge of density $+N_0 e$; then

$$\operatorname{div} \mathbf{E} = 4\pi e\,(N_0 - N_e), \qquad \operatorname{rot} \mathbf{E} = 0, \tag{5–46}$$

together with the equation linking N_e and F:

$$N_e = \int F\, d\xi. \tag{5–47}$$

[We might add the equation

$$N_e \mathbf{u}_e = \int F \boldsymbol{\xi}\, d\xi,$$

defining the mean flow velocity of the electrons and the electric current; as a consequence of (5–45) we then find that current and charge density are connected by the usual equation of continuity, but this adds nothing new to what we have in (5–45), (5–46), and (5–47)].

Equations (5–45) through (5–47) contain the greater part of the theory of electron beams, provided they are supplemented by appropriate initial conditions and boundary conditions. (The reader should consult the material given in Chapter 4 by Gould.)

In order to obtain the theory of the klystron, we suppose that there is only a component E_1 parallel to the x_1-axis ($E_2 = E_3 = 0$), and assume F to be independent of x_2 and x_3. We may then suppose that the value of F at the point $x_1 = 0$ is given by an expression of the type

$$F(\xi_1, \xi_2, \xi_3, x_1 = 0, t) = \frac{N}{\pi^{3/2} a^3} \exp \frac{-\{[\xi_1 - w(t)]^2 + \xi_2^2 + \xi_3^2\}}{a^2}. \tag{5–48}$$

Modulation effects can be obtained by making $w(t)$ of the form

$$w(t) = w_0 + w_1 \sin \omega t,$$

where w_0 is a constant, w_1 being the modulation amplitude. In many cases w_0 is made large compared with the thermal velocity a. In the usual form of the klystron, the value of N is practically a constant, but in other arrangements modulation of N is possible.

The effects of "electron bunching" come to light if we consider the characteristics of Eq. (5–45). The effects of space charge are brought into the picture by (5–46) and (5–47). When one temporarily assumes that E can be treated as a given function of x_1, a formal solution of (5–45) satisfying the initial condition (5–48) can be written. One might use this result to calculate N_e with the aid of (5–47) and substitute its value into (5–46) in order to arrive at an integro-differential equation for E_1. The resulting equation is complicated and is not linear. There appears the possibility that trajectories will cross in the x_1, t-plane; this can lead to a kind of shock wave and will cause the production of harmonics of all orders of the modulation frequency.

Another case is presented by traveling wave tubes. Here the **E**-field is the sum of two contributions, one a divergence-free field impressed from the outside, being a periodic function of, say, $\nu x_1 - \omega t$, while the other

field is connected with the electron density through Eqs. (5–46) and (5–47) as before. In this case a particular part can be played by electrons with velocities close to the wave velocity ω/ν. These electrons can be trapped between the crest and a trough of the electrostatic potential wave and there can be a strong exchange of energy between these electrons and the wave. By means of an accelerated wave it is possible to pump energy into a group of such electrons and to speed them up close to the speed of light. Conversely, if the wave is slowed down, some of the particles will lose their energy and more or less turbulent motion can result.

Hence there appear two types of nonlinearities, klystron bunching and trapping of electrons in a potential wave, which are inherently outside of the nonlinearities of hydrodynamics. Gross is of the opinion that these phenomena have not yet received the attention they deserve, and he expects that their investigation may be of importance for many fields of research, e.g. the theory of fusion processes, the physics of the solar atmosphere, and the theory of the Fermi mechanism for the acceleration of cosmic ray particles [13].

In reply to Gross' remarks, Herman Haus observed that the klystron problem actually has been solved, even with shocks [14]. Gross asked why people who are working with microwave tubes do not undertake a systematic study to understand more of these large-amplitude occurrences, since they lead to noise activity of an irregular nature and to turbulence.

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13. Reference can be made to a series of papers on plasma oscillations by D. Bohm and E. Gross, starting with: Theory of plasma oscillations. A. Origin of medium-like behavior, *Phys. Rev.* **75**, 1851–1864 (1949); B. Excitation and damping of oscillations, *ibid.*, 1864–1876.

14. Dr. Haus gave the following references:

D. L. Webster, The theory of klystron oscillations, *J. Appl. Phys.* **10**, 864–872 (1939).

D. R. Hamilton, J. K. Knipp, and J. B. H. Kuper, Klystrons and microwave triodes, *Rad. Lab. Series* **7** (McGraw-Hill, 1948), Sec. 9.3.

A. Nordsieck, Theory of the large signal behavior of traveling-wave amplifiers, *Proc. Inst. Radio Engrs.* **41**, 630–637 (1953).

H. A. Haus, Propagation of noise and signals along electron beams at microwave frequencies, Sc.D. Thesis, M.I.T. (June 1954).

F. K. Tien, L. R. Walker, and V. M. Wolontis, A large signal theory of traveling-wave amplifiers, *Proc. Inst. Radio Engrs.* **43**, 260–277 (1955).

C. G. Cutler, The nature of power saturation in traveling-wave tubes, *Bell Syst. Tech. J.* **35**, 841–876 (1956).

J. E. Rowe, A large signal analysis of the traveling-wave amplifier: theory and general results, *IRE, PGED* **ED3**: No. 1, 39–56 (1956).

S. E. Webber, Ballistic analysis of a two-cavity finite beam klystron, *IRE, PGED* **ED5**, 98–108 (1958).

Gross believes that these microwave instruments, which make possible so many forms of well-controlled experiments, will provide an excellent means for investigating some of the most intricate processes that occur in turbulence. Haus indicated some trends in electron beam research which point this way. In addition Edward Frieman mentioned work done at Princeton on exact solutions of the nonlinear equations; John Dawson found exact solutions for both running and standing waves, essentially at zero temperature [15]; and Bernstein, Greene, and Kruskal obtained exact nonlinear solutions referring to trapped particles [16]. Gross observed that the latter paper deals with the same exact nonlinear solutions as obtained earlier by Bohm and himself [13].

5–10 Flow without collisions in a magnetic field. Parker's "modified hydromagnetic equation." A fully ionized gas will be considered, of such small density that collisions can be left out of account in comparison with the effects of electric and magnetic fields (not necessarily uniform in space and constant in time).

We start from the momentum equations of Section 5–6, Eqs. (5–28) and (5–32). The right side of the latter is replaced by zero, and the terms between [ ] on the left side will be neglected, with the exception of

$$\frac{D}{Dt}(\rho_s w_{sh}) + \rho_s w_{sh}\epsilon \cong \rho_s \frac{Dw_{sh}}{Dt}$$

(omitting a term of the second degree in the diffusion velocities). We combine Eqs. (5–28) and (5–32) in such a way that they give the values of $D(u_h + w_{sh})/Dt$. Instead of subscripts s and t we use superscripts i (for the positive ions) and e (for the electrons); we also write M for the ion mass and m for the electron mass and put $\gamma = m/M$. The number density for ions and electrons is assumed to be very nearly the same: $N^i \cong N^e \cong N$. The following equations are obtained:

$$NM\frac{D}{Dt}(u_h + w_h^i) = -\frac{\partial}{\partial x_k}(p^i)_{hk} + NeE_h^* + \frac{Ne}{c}(\mathbf{w}^i \times \mathbf{B})_h; \qquad (5\text{–}49a)$$

$$Nm\frac{D}{Dt}(u_h + w_h^e) = -\frac{\partial}{\partial x_k}(p^e)_{hk} - NeE_h^* - \frac{Ne}{c}(\mathbf{w}^e \times \mathbf{B})_h, \qquad (5\text{–}49b)$$

together with $w_h^i + \gamma w_h^e = 0$. We thus have three sets of equations for the components of three vectors $\mathbf{u}$, $\mathbf{w}^i$, $\mathbf{w}^e$.

15. JOHN DAWSON, Non-linear electron oscillations in a cold plasma, *Project Matterhorn* (Princeton U.), PM-S-36, NYO-8050 (August 1958).

16. I. B. BERNSTEIN, J. M. GREENE, and M. D. KRUSKAL, Exact nonlinear plasma oscillations, *Phys. Rev.* **108**, 546–550 (1957).

Coordinate axes will be used with the x_1-axis parallel to the *local* direction of the vector $\mathbf{B}$ at the point of interest, the x_2-axis directed towards the local center of curvature of the line of force through the origin, and the x_3-axis perpendicular to x_1 and x_2. With R = radius of curvature of the line of force through the origin, and using R_2, R_3 as quantities related to the divergence of the lines of force, the following relations hold at the origin:

$$B_1 = B, \qquad B_2 = B_3 = 0,$$

$$\frac{\partial B_2}{\partial x_1} = \frac{B}{R}, \qquad \frac{\partial B_3}{\partial x_1} = 0, \qquad \frac{\partial B_2}{\partial x_2} = \frac{B}{R_2}, \qquad \frac{\partial B_3}{\partial x_3} = \frac{B}{R_3},$$

$$\frac{\partial B_1}{\partial x_1} = -\left(\frac{B}{R_2} + \frac{B}{R_3}\right).$$

The components of $\nabla \times \mathbf{B}$ are

$$\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3}, \qquad \frac{\partial B_1}{\partial x_3}, \qquad \frac{B}{R} - \frac{\partial B_1}{\partial x_2}.$$

It will be assumed that the Larmor radii for both ions and electrons are small compared with the quantities R, R_2, R_3. Since there are no collisions there can be an appreciable difference between $p_{11} = p_s$ on the one hand, and p_{22}, p_{33} on the other hand, but it is admissible to suppose $p_{22} = p_{33} = p_n$ [17a]. Simple geometrical considerations concerning stress equilibrium lead to

$$\frac{\partial p_{12}}{\partial x_1} = \frac{p_s - p_n}{R}, \qquad \frac{\partial p_{12}}{\partial x_2} = \frac{p_s - p_n}{R_2}, \qquad \frac{\partial p_{13}}{\partial x_3} = \frac{p_s - p_n}{R_3},$$

$$\frac{\partial p_{13}}{\partial x_1} = \frac{\partial p_{23}}{\partial x_2} = \frac{\partial p_{23}}{\partial x_3} = 0.$$

With the aid of these results, we find

$$\frac{\partial p_{h1}}{\partial x_h} = \frac{\partial p_s}{\partial x_1} - \frac{p_s - p_n}{B}\frac{\partial B_1}{\partial x_1}, \qquad (5\text{–}50)$$

$$\frac{\partial p_{h2}}{\partial x_h} = \frac{\partial p_n}{\partial x_2} + \frac{p_s - p_n}{B}\frac{\partial B_2}{\partial x_1}, \qquad \frac{\partial p_{h3}}{\partial x_h} = \frac{\partial p_n}{\partial x_3}.$$

These relations can be used separately for the components of the ion pressure and for those of the electron pressure.

17a. This assumption is also made by GRAD (compare Section 5–12) and by G. F. CHEW, M. L. GOLDBERGER, and F. E. LOW in the paper "The Boltzmann equation and the one-fluid hydromagnetic equations in the absence of particle collisions," *Proc. Roy. Soc.* (*London*) *A* **236**, 112–118 (1956), Eq. 20, p. 115.

After these preparations we solve Eqs. (5–49a) and (5–49b) for the directions perpendicular to the lines of the magnetic force (x_2 and x_3 components). Starting with the equation for the electrons, in which we neglect the inertia terms, and introducing the "drift vector"

$$\mathbf{U} = c\frac{\mathbf{E}\times\mathbf{B}}{B^2}, \tag{5–51}$$

we derive the following expressions:

$$w_2^e = -u_2 + U_2 + \frac{c}{NeB}\frac{\partial p_n^e}{\partial x_3},$$

$$w_3^e = -u_3 + U_3 - \frac{c}{NeB}\left(\frac{\partial p_n^e}{\partial x_2} + \frac{p_s^e - p_n^e}{R}\right).$$

These results can be combined into the formula

$$\mathbf{w}^e = -\mathbf{u} + \mathbf{U} - \frac{c}{NeB^2}\left\{(\mathbf{B}\times\nabla)p_n^e + \mathbf{B}\times[(\mathbf{B}\cdot\nabla)\mathbf{B}]\frac{p_s^e - p_n^e}{B^2}\right\}. \tag{5–52}$$

From $\mathbf{w}^e$ we obtain $\mathbf{w}^i = -\gamma\mathbf{w}^e$. We neglect, however, $\mathbf{w}^i$ in Eq. (5–49a), retaining on the other hand the inertia term. Keeping Du/Dt as a quantity to be considered later, we solve for u_2 and u_3 and combine the results into the formula

$$\mathbf{u} = \mathbf{U} + \frac{Mc}{eB^2}\left(\mathbf{B}\times\frac{D\mathbf{u}}{Dt}\right) + \\ + \frac{c}{NeB^2}\left\{(\mathbf{B}\times\nabla)p_n^i + \mathbf{B}\times[(\mathbf{B}\cdot\nabla)\mathbf{B}]\frac{p_s^i - p_n^i}{B^2}\right\}. \tag{5–53}$$

Substitution of $\mathbf{u}$ into (5–52) leads to a transformed expression for $\mathbf{w}^e$. We write at once the expression for the conduction current, $\mathbf{J} = -Ne\mathbf{w}^e$ (with sufficient approximation):

$$\mathbf{J} = \frac{NMc}{B^2}\left(\mathbf{B}\times\frac{D\mathbf{u}}{Dt}\right) + \frac{c}{B^2}\left\{(\mathbf{B}\times\nabla)p_n + \mathbf{B}\times[(\mathbf{B}\cdot\nabla)\mathbf{B}]\frac{p_s - p_n}{B^2}\right\} \tag{5–54}$$

where $p_n = p_n^i + p_n^e$, etc. Neglecting the displacement current, we now make use of the relation

$$4\pi(\mathbf{J}/c) = \nabla\times\mathbf{B}$$

in order to obtain an equation for $D\mathbf{u}/Dt$.

We apply the identity

$$\nabla\times\mathbf{B} = -\frac{\mathbf{B}\times[\mathbf{B}\times(\nabla\times\mathbf{B})]}{B^2},$$

which holds so long as we restrict to the directions of x_2 and x_3. Omitting the details of the calculation, the result is found to be

$$NM\frac{D\mathbf{u}}{Dt} = -\nabla\left(p_n + \frac{B^2}{8\pi}\right) - (\mathbf{B}\cdot\nabla)\mathbf{B}\left(\frac{p_s - p_n}{B^2} - \frac{1}{4\pi}\right). \tag{5–55}$$

With the aid of this result we can eliminate $D\mathbf{u}/Dt$ from Eq. (5–53). The result is

$$\mathbf{u} = \mathbf{U} - \frac{c}{NeB^2}\left\{(\mathbf{B}\times\nabla)\left(p_n^e + \frac{B^2}{8\pi}\right) + \right. \\ \left. + \mathbf{B}\times[(\mathbf{B}\cdot\nabla)\mathbf{B}]\left(\frac{p_s^e - p_n^e}{B^2} - \frac{1}{4\pi}\right)\right\}. \tag{5–56}$$

The equations thus obtained from the continuum equations are the same as those deduced in two papers by Parker [17b] [Eq. (5–56) given above is identical with Eq. (53) of Parker's paper of 1957; Eq. (5–55) is identical with Eq. (1) of the paper of 1958, but differs from the corresponding equation (44) in the 1957 paper, insofar as it is there given with $D\mathbf{U}/Dt$ on the left side instead of $D\mathbf{u}/Dt$ (in Parker's notation: $d\mathbf{u}_D/dt$ instead of $d\mathbf{v}/dt$ in the 1957 paper, or of $d\mathbf{u}_n/dt$ in the 1958 paper)]. Equation (5–55) is called the "modified hydromagnetic equation."

Parker deduced his results from a particle picture, resolving the motions of the ions and the electrons into various parts and deducing expressions for the separate parts (circulating current, drift current, and polarization current). Combination of these parts leads to the resulting equations for the flow of the ions and for the current.

A somewhat similar calculation, also starting from a particle picture, has been given by Brueckner and Watson [18], who write for the flow velocity:

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4,$$

17b. E. N. Parker, Newtonian development of the dynamical properties of ionized gases of low density, *Phys. Rev.* **107**, 923–933 (1957); Dynamical instability in an anisotropic ionized gas of low density, *Phys. Rev.* **109**, 1874–1876 (1958).

18. K. A. Brueckner and K. M. Watson, Use of the Boltzmann equation for the study of ionized gases of low density, *Phys. Rev.* **102**, 19–27 (1956).

with

$$\mathbf{v}_1 = \frac{c}{Ne}\left(\mathbf{B} \times \frac{\nabla \cdot \mathbf{p}}{B^2}\right), \qquad \mathbf{v}_2 = c\frac{\mathbf{E} \times \mathbf{B}}{B^2} = \mathbf{U},$$

$$\mathbf{v}_3 = \frac{Mc}{eB^2}\left(\mathbf{B} \times \frac{D\mathbf{v}_2}{Dt}\right), \qquad \mathbf{v}_4 = \frac{Mc}{eB^2}\left(\mathbf{B} \times \frac{D\mathbf{v}_1}{Dt}\right).$$

Comparison with Eq. (5–53) shows that $\mathbf{v}_1$ corresponds to the third term on the right side of (5–53), $\mathbf{v}_2$ is the first term, while $\mathbf{v}_3$ and $\mathbf{v}_4$ together give an approximation to the second term of (5–53).

In a letter to the author, Grad mentioned still another approach [19], which gave the first order result:

$$\mathbf{V} = \mathbf{V}_{||} + c\frac{\mathbf{E} \times \mathbf{B}}{B^2} \qquad (=\mathbf{V}_{||} + \mathbf{U}).$$

In second order, there must be added for the components perpendicular to the lines of magnetic force:

$$\frac{c}{eB^2}\mathbf{B} \times \left\{m\frac{d\mathbf{V}}{dt} + \frac{\mu}{B}\nabla(\tfrac{1}{2}B^2)\right\},$$

where $\mu = mV_\perp^2/2B$ is the magnetic moment of the particle. The motion in the direction of $\mathbf{B}$ is given by

$$m\mathbf{B} \cdot \frac{d\mathbf{V}}{dt} = -\frac{\mu}{B}\mathbf{B} \cdot \nabla(\tfrac{1}{2}B^2).$$

Grad interprets the particle spiraling around the line of force as a "molecule" located at the guiding center with "internal energy" μB as well as a fixed magnetic moment μ. To the lowest order this describes a one-dimensional gas, since every particle has the same ("fluid") velocity $\mathbf{U} = c(\mathbf{E} \times \mathbf{B})/B^2$ perpendicular to $\mathbf{B}$ and there is a dispersion of velocities along $\mathbf{B}$. An interesting feature of this gas is the polarization: H is not the same as B. Making the conventional macroscopic approximation yields a set of fluid equations with anisotropic stress tensor, similar to those obtained by Chew, Goldberger, and Low [17a] and by Watson [18].

5–11 Flow in the direction of the magnetic flux. The equations of the preceding section, in particular Eqs. (5–52) through (5–56), are valid only for components perpendicular to the direction of the magnetic lines of force. Hence cross multiplication with $\mathbf{B}$ of any vector in these equations entails that this vector is rotated 90° around $\mathbf{B}$.

19. H. GRAD, A guiding center fluid, Proc. Sherwood Conference on Controlled Thermonuclear Reactions, Princeton U., October 1955, TID-7503, p. 495.

The question of the flow in the direction of the magnetic flux was raised by Kenneth Watson and by Marshall Rosenbluth. It was pointed out that the equation

$$\mathbf{E} + (\mathbf{u} \times \mathbf{B})/c = 0, \tag{5–57}$$

which is often given as a basic relation for plasma dynamics, will not hold in general and one must expect a component E_1 connected with the difference in inertia of the ions and the electrons (this same point is discussed by Allis in Chapter 3). An electric field will arise as soon as differences in the motions of ions and electrons threaten to lead to the appearance of a space charge; this field adjusts itself in such a way that space charges remain as small as possible. As observed by Rosenbluth, the component E_1 in most cases will be small, unless one is dealing with frequencies comparable to the plasma frequency. Instead of introducing E_1, a sufficient approximation can usually be obtained by putting the space charge equal to zero, which means that $N^i - N^e \ll N$. This entails that the divergence of the electric current vector will be very nearly zero.

When we return to Eq. (5–56) and take the cross product with $\mathbf{B}/c$, we arrive at the relation

$$\mathbf{E}^* = \mathbf{E} + (\mathbf{u} \times \mathbf{B})/c$$
$$= \frac{1}{Ne}\nabla\left(p_n^e + \frac{B^2}{8\pi}\right) + \frac{(\mathbf{B} \cdot \nabla)\mathbf{B}}{Ne}\left(\frac{p_s^e - p_n^e}{B^2} - \frac{1}{4\pi}\right), \tag{5–58}$$

again for the components perpendicular to $\mathbf{B}$ only. The relation (5–57) thus does not hold even for these components.

As regards E_1, if we make the same approximations as before, Eqs. (5–49a), (5–49b) give the results

$$NM\frac{Du_1}{Dt} = -\frac{\partial p_s^i}{\partial x_1} + B\frac{\partial B}{\partial x_1}\frac{p_s^i - p_n^i}{B^2} + NeE_1,$$

$$0 = -\frac{\partial p_s^e}{\partial x_1} + B\frac{\partial B}{\partial x_1}\frac{p_s^e - p_n^e}{B^2} - NeE_1.$$

We may consider the second equation as giving an approximate value of E_1; substitution of this value into the first equation gives

$$NM\frac{Du_1}{Dt} = -\frac{\partial p_s}{\partial x_1} + B\frac{\partial B}{\partial x_1}\frac{p_s - p_n}{B^2}. \tag{5–59a}$$

It is possible that in many cases the value of p_s can be neglected with respect to that of p_n; we then arrive at

$$NM \frac{Du_1}{Dt} \cong - \frac{p_n}{B} \frac{\partial B}{\partial x_1}, \tag{5-59b}$$

indicating a tendency for the gas to move in the direction of decreasing magnetic field, a point often mentioned in the literature [20].

5-12 Anisotropy of the pressure tensor. Anisotropy of the pressure tensor arises when collisions are insufficient to restore approximate isotropy. The transfer equations give only partial information about the magnitude of the anisotropy. By combining Eqs. (5-29) and (5-34) so as to obtain expressions for $D(p_s)_{hk}/Dt$, omitting all terms referring to collisions and also those which contain the $(p_s)_{hkl}$, the q_{sl}, and the diffusion velocities where they occur explicitly (retaining, however, the electric currents), we arrive at

$$\frac{D(p_s)_{hk}}{Dt} = - \left\{(p_s)_{hk}\epsilon + (p_s)_{hl} \frac{\partial u_k}{\partial x_l} + (p_s)_{kl} \frac{\partial u_h}{\partial x_l}\right\} + \\ + \rho_s(G_h w_{sk} + G_k w_{sh}) + (E_h^* J_{sk} + E_k^* J_{sh}) + (M_s)_{hk}, \tag{5-60}$$

with $(M_s)_{hk}$ as given by (5-26) (in which $B_2 = B_3 = 0$ for our coordinate system).

It is more convenient to make use of the alternative equation constructed upon the principle indicated in Section 5-7, where the pressure tensor is defined with reference to the flow velocities u_{sh} of the separate constituents (ions and electrons) of the gas. Since in this description there is no diffusion velocity, nor an electric current relative to the mean flow for either the ions or the electrons, the equation has the form

$$\frac{D_s(p_s^0)_{hk}}{Dt} = - \left\{(p_s^0)_{hk} \frac{\partial u_{si}}{\partial x_i} + (p_s^0)_{hl} \frac{\partial u_{sk}}{\partial x_l} + (p_s^0)_{kl} \frac{\partial u_{sh}}{\partial x_l}\right\} + (M_s^0)_{hk}, \tag{5-61}$$

where again third-order moments and components of the heat flow have been left out, while now $(M_s^0)_{hk}$ must be defined with the aid of the $(p_s^0)_{hk}$. For simplicity we omit the subscript s in what follows.

20. This equation agrees with Grad's equation for dV/dt when we take $p_n = \frac{1}{2}\rho V^2$. The possibility that electrons can be trapped in some parts of the electric field makes the deduction of a simple equation for the motion along the lines of force less certain (remark by Eugene Parker).

Brueckner and Watson [18] assume that to a good approximation the tensor $(M^0)_{hk}$ may be considered to be equal to zero. This is the case if we suppose that $(p^0)_{12} = (p^0)_{13} = (p^0)_{23} = 0$; $(p^0)_{22} = (p^0)_{33}$. The simplest case to be considered is that where $\partial u_h/\partial x_k = 0$ for $h \neq k$ (no rotation and no shear). If $\partial u_2/\partial x_2 \neq \partial u_3/\partial x_3$, it is better to assume that $(p^0)_{23}$ is not zero; we then arrive at the following equations:

$$\begin{aligned}
\frac{D(p^0)_{11}}{Dt} &= -(p^0)_{11}\left(\epsilon + 2\frac{\partial u_1}{\partial x_1}\right), \\
\frac{D(p^0)_{22}}{Dt} &= -(p^0)_{22}\left(\epsilon + 2\frac{\partial u_2}{\partial x_2}\right) + \frac{2eB}{mc}(p^0)_{23}, \\
\frac{D(p^0)_{33}}{Dt} &= -(p^0)_{33}\left(\epsilon + 2\frac{\partial u_3}{\partial x_3}\right) - \frac{2eB}{mc}(p^0)_{23}, \\
\frac{D(p^0)_{12}}{Dt} &= \frac{D(p^0)_{13}}{Dt} = \frac{D(p^0)_{23}}{Dt} = 0.
\end{aligned} \tag{5-62}$$

The value of $(p^0)_{23}$ should be such that

$$\frac{D(p^0)_{22}}{Dt} = \frac{D(p^0)_{33}}{Dt} = -(p^0)_{22}\left(\epsilon + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right). \tag{5-62a}$$

The following special cases are of interest:

$$u_2 = u_3 = 0; \quad \begin{cases} \dfrac{D(p^0)_{11}}{Dt} = -3(p^0)_{11}\epsilon, \\ \dfrac{D(p^0)_{22}}{Dt} = \dfrac{D(p^0)_{33}}{Dt} = -(p^0)_{22}\epsilon; \end{cases} \tag{A}$$

$$u_1 = 0; \quad \frac{\partial u_2}{\partial x_2} = \frac{\partial u_3}{\partial x_3} \quad \begin{cases} \dfrac{D(p^0)_{11}}{Dt} = -(p^0)_{11}\epsilon, \\ \dfrac{D(p^0)_{22}}{Dt} = \dfrac{D(p^0)_{33}}{Dt} = -2(p^0)_{22}\epsilon. \end{cases} \tag{B}$$

Lyman Spitzer [21] introduces two different temperatures in this case: $T_{||}$ corresponding to $(p^0)_{11}$, and $T_\perp$ corresponding to $(p^0)_{22}$ and $(p^0)_{33}$. It is then found that

$$\frac{DT_{||}}{Dt} = -2T_{||}\epsilon; \qquad \frac{DT_\perp}{Dt} = -T_\perp\epsilon.$$

21. L. Spitzer, Jr., *Physics of Fully Ionized Gases*, Interscience Publishers, 1956, pp. 13–14.

This is sometimes expressed by saying that the adiabatic exponent has the value 3 for longitudinal compression, and the value 2 for transverse compression.

Grad in reference [19] had proposed the following expression for the stress tensor:

$$p_{ij} = p_s \frac{B_i B_j}{B^2} + p_n \left(\delta_{ij} - \frac{B_i B_j}{B^2} \right)$$

(also given by Chew, Goldberger, and Low in [17a]), which he used with the conventional macroscopic mass, momentum, and energy equations, assuming the heat flow to be zero, and supplementing them by Maxwell's equations and $\mathbf{E} + (\mathbf{u} \times \mathbf{B})/c = 0$. An additional equation is necessary, since two scalar unknowns, p_s and p_n, replace the ordinary pressure p. One method is to assume $p_n = \frac{1}{2}\rho \langle V_\perp^2 \rangle = BN\mu$, where $N\mu$ is the mean magnetic moment per unit volume and where $d\mu/dt = 0$. Another method is to write the entropy as

$$\eta = \tfrac{1}{2} R \ln (p_s p_n^2) - \tfrac{5}{2} R \ln \rho,$$

and add the equation $\partial\eta/\partial t + (\mathbf{u} \cdot \nabla)\eta = 0$.

In 1956 Marion Rose and Grad rederived the equations, using the moment equations. A version of the modified guided-center fluid equations was used by Rose to study a problem in shock structure [22].

Watson made the remark that when one uses as variables the quantities

$$\mu = \tfrac{1}{2} m V_\perp^2 / B \qquad \text{and} \qquad \tfrac{1}{2} m V_{||}^2 + e \int E_{||}\, ds,$$

the transport equation can be solved explicitly. However, a simple hydrodynamic description is not obtained [22a]. Work on this aspect of the equations has been carried out by a group of scientists at Princeton, but no further information was presented in the discussion.

Rosenbluth added some observations concerning the effect of the anisotropy of the particle pressure on the stability of various cases of motion. This is a subject also considered by Parker [23].

22a. Proc. of Sherwood Conference on Controlled Thermonuclear Reactions, Gatlinburg, Tenn., June 1956, TID-7520, part 2, p. 547.

22b. See K. M. WATSON, Use of the Boltzmann equation for the study of ionized gases of low density I, *Phys. Rev.* **102,** 12–19 (1956); K. A. BRUECKNER and K. M. WATSON, paper mentioned in Ref. 18; S. CHANDRASEKHAR, A. N. KAUFMAN, and K. M. WATSON, Properties of an ionized gas of low density in a magnetic field III, *Ann. phys.* **2,** 435 ff (1957); IV, *ibid.* **5,** 1–25 (1958); and *Proc. Roy. Soc.* (*London*) *A* **245,** 435 ff (1958).

23. See Ref. 17 (second paper), and also "Plasma instability in a magnetic field," *The Plasma in a Magnetic Field,* Stanford University Press, Palo Alto, Calif., 1958. Edited by R. K. M. Landshoff.

5–13 Electric conductivity and heat flow. We shall now briefly indicate how the transfer equations can be used to calculate the electric conductivity of a completely ionized and neutral gas. In this case the effect of the collisions is of prime importance and we must use the complete equations with their right sides, as given in Section 5–6. The calculations to be carried out are analogous to those given in reference [1] for the determination of the coefficient of diffusion in a binary mixture, to what Chapman and Cowling call the "second approximation," in which account is taken of the effect of heat flow and thermal diffusion. Along with the electric field a magnetic field will be introduced. It would be worth while to have similar calculations to the second approximation for a partially ionized gas, but so far only the first approximation has been worked out [24].

In order to examine to what extent the assumption of binary collisions can be used for particles influencing each other through coulomb forces we introduce the dimensionless parameter

$$\lambda = \frac{\kappa T}{e^2} \frac{1}{N_e^{1/3}}, \tag{5–63a}$$

where $N_e = \sum_s N_s e_s^2/e^2$. This parameter is a measure for the ratio of the mean distance between charged particles and the distance at which coulomb forces produce large deflections. With the aid of this parameter the Debye shielding distance r_D defined in Eq. (5–20) can be expressed as follows:

$$(4\pi)^{1/2} r_D = \frac{e^2}{\kappa T} \lambda^{3/2} = N_e^{-1/3} \lambda^{1/2},$$

while the mean free path (apart from a numerical factor which is defined in various ways) is given by

$$2(2\pi)^{1/2} l = \frac{e^2}{\kappa T} \frac{\lambda^3}{\ln \Lambda} = N_e^{-1/3} \frac{\lambda^2}{\ln \Lambda}.$$

The quantity Λ has been defined by (5–21); it can also be written

$$\Lambda = (\tfrac{3}{2}\sqrt{\pi})\, \lambda^{3/2}. \tag{5–63b}$$

24. See T. G. COWLING, The electrical conductivity of an ionized gas in a magnetic field, with applications to the solar atmosphere and the ionosphere, *Proc. Roy. Soc.* (*London*) *A* **183,** 453–470 (1945), in particular pp. 464–466; and J. M. BURGERS, in Seminar on Applied Mathematics, Boulder, Colorado, 1957 (to be published by the Am. Math. Soc.). Since this was written the second approximation for a partially ionized gas has been fully worked out by A. C. PIPKIN, The electric conductivity of a partially ionized gas, Inst. for Fluid Dynamics and Applied Mathematics, U. of Maryland, Technical Note BN-170 (April 1959).

It follows that a large value of the parameter λ suffices to make r_D and l large compared both with $e^2/\kappa T$ and $N_e^{-1/3}$, while at the same time it leads to a large value for Λ. Under such circumstances the assumption is justified that the collision processes are mainly binary. In order that the condition shall be satisfied we must have

$$T \gg (e^2/\kappa)N_e^{1/3} = 1.67 \times 10^{-3}N_e^{1/3}.$$

When magnetic fields are present we must also require that the Larmor radius for an electron (which is smaller than that for an ion) shall be large compared with $e^2/\kappa T$; this introduces the additional condition

$$B \ll (8/\pi)^{1/2}(\kappa T/e^2)^{3/2}cm_e^{1/2} \quad \text{or} \quad B \ll 20T^{3/2}.$$

We observe that when

$$B < 4\sqrt{2}\, c(m_eN_e)^{1/2} = 5.1 \times 10^{-3}N_e^{1/2},$$

the Larmor radius will also be larger than the Debye shielding distance, but it does not appear that this is necessary.

The equations of motion for the ions and the electrons are obtained from Eq. (5-32). They are each other's opposites, so that the equations for the electrons are sufficient. The equations for the heat flow follow from (5–35); these must be written out both for the ions and for the electrons. The magnetic field will be considered as constant in time and uniform in space; the x_1-axis is in the direction of the magnetic flux, so that $B_1 = B$; $B_2 = B_3 = 0$. Also $\rho_e = \rho_e^i + \rho_e^e = 0$.

With $m_e/M = \gamma$ as before, we have $\mathbf{w}^i = -\gamma\mathbf{w}^e$. It is convenient to write

$$\mathbf{w}^i - \mathbf{w}^e = \mathbf{w}, \qquad \mathbf{J}^i + \mathbf{J}^e = \mathbf{J} = N\mathbf{e}\mathbf{w}.$$

We write K for K_{12} (1, ions; 2, electrons) as defined in (5–25), so that with $N^i = N^e$ (or $N_1 = N_2$) $= N$:

$$K = \frac{2}{3}(2\kappa T m_e)^{1/2}N^2Z = \frac{16\sqrt{\pi}}{3}\left(\frac{m_e}{2\kappa T}\right)^{3/2}\frac{N^2e^4}{m_e}\ln\Lambda. \qquad (5\text{–}64)$$

The following abbreviations will be used for quantities which often reappear:

$$\vartheta = \frac{N\kappa}{K}\nabla T, \qquad \tilde{\omega} = \frac{NeB}{Kc},$$

$$\kappa_1 = (2\sqrt{2}/5)\gamma^{-1/2}, \qquad \xi = (8 + 13\sqrt{2})/10\sqrt{2} = 1.865.$$

With these notations we arrive at the following equations (three equations of motion and two times three equations for the heat flow):

$$\frac{\partial p^e}{\partial x_1} + NeE_1 = K(w_1 - z\gamma r_1^i + zr_1^e),$$

$$\frac{\partial p^e}{\partial x_2} + NeE_2 + \frac{NeB}{c}(u_3 - w_3) = K(w_2 - z\gamma r_2^i + zr_2^e), \qquad (5\text{–}65)$$

$$\frac{\partial p^e}{\partial x_3} + NeE_3 - \frac{NeB}{c}(u_2 - w_2) = K(w_3 - z\gamma r_3^i + zr_3^e);$$

$$\begin{aligned}
\tfrac{5}{2}\vartheta_1 &= \tfrac{5}{2}z\gamma w_1 - \kappa_1 r_1^i + \tfrac{27}{10}\gamma r_1^e,\\
\tfrac{5}{2}\vartheta_2 - \tilde{\omega}r_3^i &= \tfrac{5}{2}z\gamma w_2 - \kappa_1 r_2^1 + \tfrac{27}{10}\gamma r_2^e,\\
\tfrac{5}{2}\vartheta_3 + \tilde{\omega}r_2^i &= \tfrac{5}{2}z\gamma w_3 - \kappa_1 r_3^i + \tfrac{27}{10}\gamma r_3^e;
\end{aligned} \qquad (5\text{–}66a)$$

$$\begin{aligned}
\tfrac{5}{2}\vartheta_1 &= -\tfrac{5}{2}zw_1 + \tfrac{27}{10}\gamma r_1^i - \xi r_1^e,\\
\tfrac{5}{2}\vartheta_2 + \tilde{\omega}r_3^e &= -\tfrac{5}{2}zw_2 + \tfrac{27}{10}\gamma r_2^i - \xi r_2^e,\\
\tfrac{5}{2}\vartheta_3 - \tilde{\omega}r_2^e &= -\tfrac{5}{2}zw_3 + \tfrac{27}{10}\gamma r_3^i - \xi r_3^e.
\end{aligned} \qquad (5\text{–}66b)$$

The coefficient z, defined in (5–23), has the value $\frac{3}{5}$ for coulomb forces.

5-14 Electric conductivity and heat flow. (Cont.) To work out the conductivity problem we take $\nabla p = 0$, $\nabla T = 0$ ($\vartheta = 0$). We pick out first the equations referring to the x_1-direction:

$$\begin{aligned}
NeE_1 &= K(w_1 - z\gamma r_1^i + zr_1^e),\\
0 &= \tfrac{5}{2}z\gamma w_1 - \kappa_1 r_1^i + \tfrac{27}{10}r_1^e,\\
0 &= -\tfrac{5}{2}zw_1 + \tfrac{27}{10}\gamma r_1^i - \xi r_1^e.
\end{aligned} \qquad (5\text{–}67)$$

We can expect that $\gamma r_1^i \ll r_1^e$ (which is confirmed by the solution). We then find

$$r_1^e = -\frac{5}{2}\frac{z}{\xi}w_1 \qquad \text{and} \qquad w_1 = \frac{Ne}{K}\frac{1}{1-\Delta}E_1,$$

where $\Delta = \frac{5}{2}(z^2/\xi) = 0.483$, so that $1/(1 - \Delta) = 1.93$. Keeping in mind that the electric current is given by $\mathbf{J} = N\mathbf{e}\mathbf{w}$, it is convenient to write

$$\sigma_0 = \frac{N^2e^2}{K} = \frac{3}{16\sqrt{\pi}}\left(\frac{2\kappa T}{m_e}\right)^{3/2}\frac{m_e}{e^2}\frac{1}{\ln\Lambda}. \qquad (5\text{–}68)$$

In electrostatic units: $\sigma_0 = 6.9 \times 10^7 T^{3/2}/\ln \Lambda;$

in electromagnetic units: $\sigma_0 = 7.7 \times 10^{-14} T^{3/2}/\ln \Lambda,$

or $\sigma_0 = 7.7 \times 10^{-5} T^{3/2}/\ln \Lambda$ mho/cm.

We then have [by (5–18)]:

$$J_1 = \frac{\sigma_0}{1 - \Delta} E_1 = 1.93 \sigma_0 E_1. \tag{5–69}$$

In treating the remaining six equations, referring to the x_2- and x_3-directions, we assume that γr_2^i and γr_3^i will be small compared with r_2^e and r_3^e. The last two equations of (5–66b) then reduce to

$$\xi r_2^e + \tilde{\omega} r_3^e = -\tfrac{5}{2} z w_2,$$

$$-\tilde{\omega} r_2^e + \xi r_3^e = -\tfrac{5}{2} z w_3,$$

from which we can obtain r_2^e and r_3^e in terms of w_2 and w_3. Substitution of the results into the second and third equations of (5–66a) gives two equations for the calculation of r_2^i and r_3^i, confirming the smallness of γr_2^i, γr_3^i.

We now write the second and third equation of the system (5–65) in the form

$$NeE_2^*/K = w_2 + \tilde{\omega} w_3 + z r_2^e,$$

$$NeE_3^*/K = -\tilde{\omega} w_2 + w_3 + z r_3^e,$$

and substitute the values of r_2^e, r_3^e. The resulting equations can then be solved for w_2 and w_3. The corresponding components of the electric current are found to be

$$J_2 = \sigma_0 \frac{E_2^*(1 - \delta) - E_3^* \tilde{\omega}(1 + \delta/\xi)}{(1 - \delta)^2 + \tilde{\omega}^2 (1 + \delta/\xi)^2},$$

$$J_3 = \sigma_0 \frac{E_2^* \tilde{\omega}(1 + \delta/\xi) + E_3^*(1 - \delta)}{(1 - \delta)^2 + \tilde{\omega}^2 (1 + \delta/\xi)^2}, \tag{5–70a}$$

where

$$\delta = \frac{5}{2} z^2 \frac{\xi}{\xi^2 + \tilde{\omega}^2} < \Delta. \tag{5–70b}$$

The results thus obtained are in satisfactory agreement with those given by Spitzer [25] and by Landshoff [26], which have been deduced according to a different method [27].

It is of interest to invert Eqs. (5–69) and (5–70); we find

$$E_1^* = E_1 = (1 - \Delta) J_1/\sigma_0,$$

$$E_2^* = \{(1 - \delta) J_2 + \tilde{\omega}(1 + \delta/\xi) J_3\}/\sigma_0, \tag{5–71}$$

$$E_3^* = \{-\tilde{\omega}(1 + \delta/\xi) J_2 + (1 - \delta) J_3\}/\sigma_0,$$

and also

$$E_1^* J_1 + E_2^* J_2 + E_3^* J_3 = \frac{(1 - \Delta) J_1^2 + (1 - \delta)(J_2^2 + J_3^2)}{\sigma_0}. \tag{5–72}$$

The latter formula shows that the dissipation connected with the current components perpendicular to the magnetic field is somewhat larger than the dissipation connected with the component parallel to the field.

In certain cases of a field of limited extent in which an electric force E_2 is set up by exterior means, it can happen that there will arise a component E_3 as a result of polarization due to charges on the planes limiting the field,

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25. The result is in close agreement with that given by Cowling in the paper mentioned in Ref. 24.

Compare L. Spitzer, Jr., Ref. 21, p. 84, Eq. 5–37, and also L. Spitzer and R. Härm, Transport phenomena in a completely ionized gas, *Phys. Rev.* **89**, 977–981 (1953).

26. R. K. Landshoff, Transport phenomena in a completely ionized gas in presence of a magnetic field, *Phys. Rev.* **76**, 904–909 (1949). The conductivity is given here as a complex quantity, the real part being the coefficient of E_2^* in the expression for J_2, while the imaginary part is the coefficient of E_3^*. From Landshoff's Table I (*loc. cit.*, p. 906) one must take the values of Δ_∞/Δ given for $Z = 1$ (singly charged ions); the parameter ω/ν is the same as our $\tilde{\omega}$ (with the correction of an error of print in Landshoff's Equation 62, p. 908, where apparently a factor $\sqrt{\pi}$ has been lost).

27. With collision cross sections such as hold for coulomb forces there is no difference in cross section for ion-ion, ion-electron, or electron-electron collisions. Whereas the mass of the electron in most instances can be neglected in comparison with the mass of the ion, we do not have before us the case of a Lorentz gas, which is characterized by the condition that one type of particle has both a very small mass and a very small collision cross section. Calculations which take the Lorentz gas as a starting point need a correction for the effect of electron-electron encounters, to which reference is made in various papers (see, for instance, Ref. 21, pp. 83–84). In the present treatment, electron-electron interactions come in through the quantities r and the coefficient κ_1, and no further correction is necessary.

of such magnitude that the current component J_3 is reduced to zero. The value of E_3 necessary to produce this result is easily found from the last line of (5–71). When this is introduced into the expression for J_2 [first line of Eq. (5–70a)], we obtain

$$J_2 = \frac{\sigma_0}{1 - \delta} E_2^*, \qquad \text{for} \qquad J_3 = 0. \tag{5–73}$$

It follows that the conductivity in the direction of E_2 now is only slightly smaller than that in the direction parallel to the magnetic field. This point was brought forward in the discussion by Sydney Chapman, who mentioned that in 1933 Cowling had already directed attention to it [28].

The expressions obtained for r_1^e, r_2^e, r_3^e (and, if desired, also those for r_1^i, etc.) can be applied in order to calculate the heat flow connected with the electric current.

Appendix to Section 14. In the discussion Chapman, as an example of the effect of the anisotropy of electric conductivity, mentioned the application to the ionosphere, which is traversed by the magnetic field of the Earth in a way strongly dependent upon latitude. The magnetic lines of force cross the layers of the ionosphere at angles varying from 90° at the magnetic poles to 0° at the magnetic equator. Electromotive forces are induced in this layer by the motion of the air through the magnetic field. But the ionosphere cannot continuously carry a current perpendicular to itself, because at its lower limit it is bordered by nonconducting gas. This condition of noncontinuous vertical flow of current produces a variation in the effective conductivity in the two horizontal directions. When horizontal coordinates are introduced, x directed to the north and y to the east, one can write the equations

$$J_x = \sigma_{xx} E_x^* + \sigma_{xy} E_y^*,$$
$$J_y = -\sigma_{xy} E_x^* + \sigma_{yy} E_y^*,$$

in which three coefficients of conductivity must be used which differ from those for either the longitudinal or the transverse electric conductivity of a gas which is not limited in any direction. The coefficients will vary with latitude, according to the inclination of the magnetic field.

This has a remarkable consequence for the electric current induced by the regular daily circulation of the atmosphere. Consider the pattern of currents which will be seen by an observer looking from the Sun toward the Earth, assuming that the Sun is in the plane of the magnetic equator. There is a narrow strip along the magnetic equator, only about 5° wide in latitude, within which the magnetic dip is small and in which the conductivity is higher than elsewhere. Here a considerable enhancement of the eastward electric current is obtained during the daylight hours. This approximately doubles the daily magnetic variation of the north component of force at equatorial stations under the strip. The effect was first observed at the observatory Huancayo in Peru, which lies almost exactly on the magnetic equator [29].

28. T. G. Cowling, The electric conductivity of an ionised gas in the presence of a magnetic field, *Monthly Notices Roy. Astron. Soc.* **93,** 90–98 (1933), in particular p. 96.

5–15 The "runaway" phenomenon. The results obtained in Section 5–14 are based upon the development of the distribution function given by (5–3) in conjunction with (5–12) through (5–14), with the assumption that only terms linear in the coefficients A_{sh}, etc. need be retained. This has led to a resistance proportional to the current and does not show the decrease of the resistance appearing when very high velocities are reached. This decrease of the resistance leads to the phenomenon of "runaway," which has turned up in many discussions [30].

We shall briefly indicate the link between the linear resistance law for low speeds and the decreasing resistance for high speeds. Instead of the development of the distribution function used before, it is convenient to assume

$$F_s = \frac{N_s}{\pi^{3/2} a_s^3} \exp \frac{-(\xi_s - \mathbf{u}_s)^2}{a_s^2}, \tag{5–74a}$$

so that the flow velocity $\mathbf{u}_s$ of each separate constituent of the gas is introduced explicitly. The mean mass flow velocity and the diffusion velocities will not be used in this treatment. For greater generality, we will not suppose that all temperatures are the same, and we will replace (5–12) temporarily by

$$a_s^2 = \frac{2\kappa T_s}{m_s}. \tag{5–74b}$$

The new expressions for the distribution functions are more suitable when the differences in velocity and in temperature are high. They neglect, however, the anisotropy which must be introduced when it is desirable to consider the tensor character of the pressure and to take account of

29. The subject has been worked upon by Cowling in the papers mentioned in Refs. 28 and 24. Also by the Australian physicists D. F. Martyn and W. G. Baker: see *Nature* **162,** 142–143 (1948); **163,** 685–686 (1949); **170,** 1090–1092 (1952); *Phil. Trans. Roy. Soc. (London) A* **246,** 281–320 (1953). S. Chapman published a paper on it under the title "The electric conductivity of the ionosphere: a review" in the *Nuovo cimento, Suppl.* **4** (ser. X), 1385–1412 (1956).

30. See remarks by James Tuck in Section 1–4 and by William Allis in Section 3–4.

heat flow. It is possible to extend form (5–74a) by multiplying it with a factor analogous to the one introduced in (5–3), but this will lead to greater complication in the integrals.

We again work out the integral for the gain of momentum in collisions:

$$\int d\xi_s \left(\frac{dF_s}{dt}\right)_{\text{coll}} m_s \xi_s,$$

and the integral for the transfer of kinetic energy:

$$\int d\xi_s \left(\frac{dF}{dt}\right)_{\text{coll}} \tfrac{1}{2} m_s (\xi_s - \mathbf{u}_s)^2.$$

These integrals will give the right-hand sides to be used in Eqs. (5–41) and (5–42). The value of $(dF_s/dt)_{\text{coll}}$ must be taken from Eq. (5–2). It is more convenient, however, to work out the completely equivalent formulas [31]:

$$\iiint d\xi_s \, d\xi_t m_s (\xi_s' - \xi_s) F_s F_t \, g \, b \, db \, d\epsilon$$

and

$$\iiint d\xi_s \, d\xi_t \tfrac{1}{2} m_s \{(\xi_s' - \mathbf{u}_s)^2 - (\xi_s - \mathbf{u}_s)^2\} F_s F_t \, g \, b \, db \, d\epsilon,$$

where ξ_s' is the value of ξ_s after the collision. We omit the details of the integrations, which can be carried out when the collision cross sections $S^{(1)}$ and $S^{(2)}$ have the values holding for coulomb forces [see formulas (22a)]. With cross sections such as are used for other types of particles (rigid spheres, for example) the integrations are more difficult. The results are

gain of momentum in collisions:

$$-\frac{4\pi N_s N_t e_s^2 e_t^2 \ln \Lambda}{\mu} \frac{\mathbf{w}}{w^3} \left\{\operatorname{erf} \frac{w}{\alpha} - \frac{2}{\sqrt{\pi}} \frac{w}{\alpha} \exp\left(\frac{-w^2}{\alpha^2}\right)\right\}; \quad (5\text{–}75a)$$

gain of kinetic energy:

$$-\frac{4\pi N_s N_t e_s^2 e_t^2 \ln \Lambda}{\mu} \frac{1}{w}\left[\sigma \operatorname{erf} \frac{w}{\alpha} - (x + \sigma)\left\{\operatorname{erf} \frac{w}{\alpha} - \frac{2}{\sqrt{\pi}} \frac{w}{\alpha} \exp\left(\frac{-w^2}{\alpha^2}\right)\right\}\right]. \quad (5\text{–}75b)$$

31. See Ref. 1, p. 66 (Sec. 3.53).

The following abbreviations have been used:

$$\mathbf{w} = \mathbf{u}_s - \mathbf{u}_t, \qquad \alpha^2 = a_s^2 + a_t^2 = \frac{2\kappa T_s}{m_s} + \frac{2\kappa T_t}{m_t},$$

$$\sigma = \frac{2\kappa}{m_0 \alpha^2}(T_s - T_t), \quad x = m_t/m_0, \quad m_0 = m_s + m_t, \quad \mu = m_s m_t / m_0.$$

After summation with respect to t (in which the case $t = s$ can be omitted), (5–75a) gives the appropriate right side for Eq. (5–41), and (5–75b) for Eq. (5–42).

Formula (5–75a) reduces to our former expression $-K\mathbf{w}$ [occurring in Eq. (5–32)] when w/α is small and $T_s = T_t$. On the other hand, when w/α is large, we can use the approximation

$$-\frac{4\pi N_s N_t e_s^2 e_t^2 \ln \Lambda}{\mu} \frac{\mathbf{w}}{w^3},$$

which is the expression mentioned by James Tuck.

As observed, the calculation is based upon the expression for the collision cross sections holding when the forces between the particles are exclusively coulomb forces. Thus the results hold strictly only for protons and electrons. Other ions, unless they are completely stripped of electrons, will exert forces of a different type at close encounters, and it must be expected that the cross section then will not go down to zero when the velocity becomes higher and higher. This point is stressed by Allis [32].

5–16 Plasma diffusion in a magnetic field. The system of equations obtained in Section 5–13 can also be used for the treatment of a problem considered by Rosenbluth and Kaufman under the title "Plasma Diffusion in a Magnetic Field" [33]. What they call "plasma diffusion" is considered here as "flow," since it is the motion of ions and electrons together (in a similar way as is the case with the so-called ambipolar diffusion), not a diffusion of one species relative to the other. The authors treated the problem by starting from the equation for the distribution functions.

A fully ionized plasma is considered, in a magnetic field constant in time. We take the x_1-axis in the direction of the magnetic flux $\mathbf{B}$. This flux, although having the same direction everywhere in the field, is not supposed to be uniform, but is a function of x_2. Initially, at $t = 0$, B is large for $x_2 > 0$ and small or zero for $x_2 < 0$, and it is assumed that the pressure field in the gas is adjusted in such a way that it balances the difference of the magnetic pressures on both sides of the plane $x_2 = 0$.

32. See Section 3–4.

33. M. N. Rosenbluth and A. N. Kaufman, Plasma diffusion in a magnetic field, *Phys. Rev.* **109**, 1–5 (1958).

This necessitates an electric current in the x_3-direction, flowing in a thin sheet at $x_2 = 0$. In consequence of the thermal motion of the ions and the electrons, both types of particles will escape towards the region $x_2 > 0$; this escape phenomenon is studied under the name of "plasma diffusion." Since one expects that no large space charges can appear in the field, the local mean density of ions and electrons must always be very nearly the same, and this implies that the mean speed of flow of the ions and of the electrons must have nearly the same value. Hence there will be no electric current in the x_2-direction and $w_2 = 0$, while $u_2 \neq 0$. At the same time there will appear an electric field E_2 of such magnitude that the equality of flow velocities of ions and electrons is assured. There may also be a temperature field, which is supposed to be a function of x_2, as is the case with the pressure and with B. There may also be electric force components E_1 and E_3 influencing the system of electric currents.

The flow problem can be treated with Eqs. (5–65), (5–66a), and (5–66b), in which we retain $\partial p^e/\partial x_2$ and ϑ_2. We assume that u_1 and u_3 are zero with respect to the system of coordinates introduced. The number density of both the ions and electrons is indicated by N. We also suppose that ions and electrons have the same temperature. Then

$$p^i = p^e = N\kappa T = \tfrac{1}{2}p.$$

We must supplement the system (5–65), (5–66a), and (5–66b) by the equation for the mass flow (5–28) and the continuity equations. In Eq. (5–28), the only component of importance is that corresponding to the x_2-direction. Taking $\mathbf{G} = 0$, $\rho_e = 0$, and neglecting the inertia term, it reduces to

$$\frac{\partial p}{\partial x_2} - \frac{J_3 B}{c} = 0,$$

from which we have

$$J_3 = New_3 = \frac{c}{B}\frac{\partial p}{\partial x_2} = \frac{2c}{B}\frac{\partial p^e}{\partial x_2}. \tag{5–76}$$

The equations referring to the x_1-direction are the same as those used in Section 5–14; thus, again,

$$J_1 = New_1 = \frac{N^2e^2}{K}\frac{E_1}{1-\Delta} = \frac{\sigma_0}{1-\Delta}E_1. \tag{5–77}$$

The remaining equations have the form:

from (5–65)
$$\begin{cases} \dfrac{\partial p^e}{\partial x_2} + NeE_2 - \dfrac{NeB}{c}w_3 = K(\quad - z\gamma r_2^i + zr_2^e), \\[2ex] NeE_3 - \dfrac{NeB}{c}u_2 = K(w_3 - z\gamma r_3^i + zr_3^e); \end{cases}$$

from (5–66a)
$$\begin{cases} \dfrac{5}{2}\vartheta_2 - \tilde{\omega}r_3^i = \quad - \kappa_1 r_2^i + \dfrac{27}{10}\gamma r_2^e, \\[2ex] \tilde{\omega}r_2^i = \dfrac{5}{2}z\gamma w_3 - \kappa_1 r_3^i + \dfrac{27}{10}\gamma r_3^e; \end{cases}$$

from (5–66b)
$$\begin{cases} \dfrac{5}{2}\vartheta_2 + \tilde{\omega}r_3^e = \quad \dfrac{27}{10}\gamma r_2^i - \xi r_2^e, \\[2ex] -\tilde{\omega}r_2^e = -\dfrac{5}{2}zw_3 + \dfrac{27}{10}\gamma r_3^i - \xi r_3^e. \end{cases}$$

Rosenbluth and Kaufman suppose that the Larmor frequency of both the electrons and the ions is large compared with their respective collision frequencies. This requires that not only $\tilde{\omega}$, but even $\tilde{\omega}\sqrt{\gamma}$ must be large compared with unity; thus $\tilde{\omega} \gg \kappa_1$.

In handling these equations we again assume that γr_2^i and γr_3^i are small compared with r_2^e, r_3^e; we can then neglect the first-mentioned quantities in the fifth and sixth equations. Solving for r_2^e, r_3^e, we obtain

$$r_2^e \cong \frac{5}{2}z\frac{w_3}{\tilde{\omega}} - \frac{5}{2}\frac{\xi\vartheta_2}{\tilde{\omega}^2}, \qquad r_3^e \cong -\frac{5}{2}z\frac{\xi w_3}{\tilde{\omega}^2} - \frac{5}{2}\frac{\vartheta_2}{\tilde{\omega}}.$$

We next use the third and fourth equations to obtain approximate expressions for r_2^i, r_3^i:

$$r_2^i \cong -\frac{5}{2}\frac{\kappa_1\vartheta_2}{\tilde{\omega}^2}, \qquad r_3^i \cong \frac{5}{2}\frac{\vartheta_2}{\tilde{\omega}}.$$

The sum

$$r_2^i + r_2^e \cong -\frac{5}{2}\frac{\kappa_1\vartheta_2}{\tilde{\omega}^2} + \frac{3}{2}\frac{w_3}{\tilde{\omega}} \tag{5–78}$$

can be used to find the heat flow in the direction of x_2:

$$q_2 = q_2^* = N\kappa T(r_2^i + r_3^i).$$

Turning to the first equation, we find

$$\frac{1}{K}\frac{\partial p^e}{\partial x_2} + \frac{NeE_2}{K} = \tilde{\omega}w_3 + zr_2^e \cong \tilde{\omega}w_3.$$

This gives the electric field component E_2, which ensures equality of flow of ions and electrons:

$$E_2 = -\frac{1}{Ne}\frac{\partial p^e}{\partial x_2} + \frac{Bw_3}{c} = \frac{1}{2Ne}\frac{\partial p}{\partial x_2}. \tag{5–79}$$

Finally, the second equation of the system takes the form

$$\frac{NeE_3}{K} - \tilde{\omega}u_2 = w_3 + zr_3^e \cong w_3 - \frac{3}{2}\frac{\vartheta_2}{\tilde{\omega}},$$

from which we obtain

$$u_2 = \frac{cE_3}{B} - \frac{w_3}{\tilde{\omega}} + \frac{3}{2}\frac{\vartheta_2}{\tilde{\omega}^2}, \tag{5–80a}$$

or, when use is made of the expressions for w_3 and for $\tilde{\omega}$ [34],

$$u_2 = \frac{cE_3}{B} - \frac{1}{\sigma_0}\frac{c^2}{B^2}\frac{\partial p}{\partial x_2} + \frac{3}{2\sigma_0}\frac{c^2}{B^2}N\frac{\partial(\kappa T)}{\partial x_2}. \tag{5–80b}$$

The results obtained in this way from the transfer equations are essentially the same as those given by Rosenbluth and Kaufman. Account should be taken of the difference in notation; moreover, Rosenbluth and Kaufman take $E_1 = E_3 = 0$ in their final results [an error of print occurs in their Eq. (10′); when this is corrected, our Eq. (5–78) leads to their Eq. (15); our (5–79) to their (34); and our (5–80a) or (5–80b) to their (10)].

Rosenbluth and Kaufman have added the equations of continuity and the energy equation in order to obtain a complete system, from which the behavior of u_2 as a function of x_2 and t can be derived.

5–17 Phenomena of thermal diffusion. In the discussion Chapman called attention to the phenomena of thermal diffusion which are connected with the occurrence of a thermal gradient in a gas.

Taking first the most simple case, a binary mixture when there is no pressure gradient, no gravity, and no electric or magnetic field, Eqs. (5–65), (5–66a), and (5–66b), written in vector form, reduce to

$$\begin{aligned} 0 &= \mathbf{w} - z\gamma\mathbf{r}^i + z\mathbf{r}^e, \\ (5/2)\boldsymbol{\vartheta} &= (5z/2)\gamma\mathbf{w} - \kappa_1\mathbf{r}^i + (27/10)\gamma\mathbf{r}^e, \\ (5/2)\boldsymbol{\vartheta} &= -(5z/2)\mathbf{w} + (27/10)\gamma\mathbf{r}^i - \xi\mathbf{r}^e. \end{aligned}$$

Bearing in mind the smallness of γ, we find

$$\mathbf{w} = \frac{3/2}{\xi - 9/10}\boldsymbol{\vartheta} = 1.55\boldsymbol{\vartheta},$$

$$\mathbf{r}^i = -\frac{25\sqrt{\gamma}}{4\sqrt{2}}\boldsymbol{\vartheta}, \qquad \mathbf{r}^e = -\frac{5/2}{\xi - 9/10}\boldsymbol{\vartheta} = -2.59\boldsymbol{\vartheta}.$$

34. The second term of Eq. (5–80b) is essentially the same as what is given by SPITZER, Ref. 21, p. 38, Eq. (3–16). Equation (5–76) above is the same as Spitzer's equation (2–24), p. 24 of Ref. 21.

Since $\mathbf{w}^i = \gamma\mathbf{w}$, and $\mathbf{w}^e \cong -\mathbf{w}$, it follows that the ions tend to diffuse in the direction of increasing temperature and the electrons in the direction of decreasing temperature.

Actually, an electric field will appear which counteracts this process and only an extremely slight separation will occur. We can consider the equilibrium situation in which the state of the system is independent of the time and where all diffusion velocities $\mathbf{w}_s$ are zero. The effect of the temperature is then balanced by a pressure gradient or by an electric field. It will be assumed that there is no magnetic field. Keeping to a binary mixture, Eqs. (5–65), (5–66a), and (5–66b) now give

$$\begin{aligned} \nabla p^e + NeE &= K(-z\gamma\mathbf{r}^i + z\mathbf{r}^e), \\ (5/2)\boldsymbol{\vartheta} &= -\kappa_1\mathbf{r}^i + (27/10)\gamma\mathbf{r}^e, \\ (5/2)\boldsymbol{\vartheta} &= (27/10)\gamma\mathbf{r}^i - \xi\mathbf{r}^e. \end{aligned}$$

From the second and third equations, we obtain

$$\mathbf{r}^i = -\frac{25\sqrt{\gamma}}{4\sqrt{2}}\boldsymbol{\vartheta}, \qquad \mathbf{r}^e = -\frac{5}{2\xi}\boldsymbol{\vartheta},$$

after which the first equation yields

$$\nabla p^e + NeE = -(3/2\xi)N\kappa\nabla T.$$

Since the gas as a whole can be treated as neutral, Eq. (5–28) gives $\nabla p = \rho\mathbf{G}$, from which $\nabla p^i = \nabla p^e = \frac{1}{2}\nabla p = \frac{1}{2}\rho\mathbf{G}$. Making use of this result, we arrive at

$$NeE = -\tfrac{1}{2}\rho\mathbf{G} - (3/2\xi)N\kappa\nabla T. \tag{5–81a}$$

Here $\mathbf{G}$ is the acceleration of gravity, defined as a vector. When the z-axis of the coordinate system is taken vertically upward while gravitation is acting downward, we obtain

$$NeE_z = \tfrac{1}{2}\rho|G| - (3/2\xi)N\kappa(dT/dz). \tag{5–81b}$$

In a case without gravity the electric field is in the direction of decreasing temperature; it must prevent the ions from going to the region of higher temperature. When there is a gravity field, part of the electric field must serve to hold down the electrons and to pull up the ions against gravity; this part supports about half the weight of the ions if the ions are singly ionized, as was supposed here.

A more complicated case is encountered when the ionized gas contains positive ions of a different type. Chapman referred to conditions in the solar corona and supposed that the gas, consisting for the major part of

protons and electrons, would also contain a small amount of positive ions of atomic weight A, much larger than unity, stripped of a great number of electrons, so that the charge of these ions would be ne, with n possibly of the order 10 to 15. Ions of calcium ($A = 40$), having lost 14 electrons, and of nickel ($A = 57$), having lost 15 electrons, are typical examples, but ions of iron ($A = 56$) and of argon ($A = 40$) that have lost a smaller number of electrons may also come into this category. The number of these ions per unit volume, N_3, is supposed to be so small that we can still assume that the number density of the protons is the same as that of the electrons (N for both).

Again considering a case of equilibrium with all diffusion velocities equal to zero, we now write the heat flow equations (5–35) for the protons (subscript $s = 1$), the electrons ($s = 2$), and for the heavy ions ($s = 3$), taking into account the ratios between the masses, the cross sections, and the number densities, all of which influence the ratios between the various coefficients K_{st}. Since collision cross sections for coulomb forces are proportional to the square of the charges of both components, there appears a factor n^2 in K_{13} and in K_{23}, and a factor n^4 in K_{33}.

Omitting the details of the calculations, in which many terms appear to be negligible, either in consequence of the smallness of m_2/m_1 and m_2/m_3, or that of N_3/N, we arrive at the following results:

(a) $$\mathbf{r}_1 = -\frac{25\sqrt{\gamma}}{4\sqrt{2}}\,\vartheta, \qquad \mathbf{r}_2 = -\frac{5}{2\xi}\,\vartheta \qquad \text{(the same as before)},$$

(b) $$\mathbf{r}_3 = -\left(\frac{45}{8\sqrt{2}}\,\frac{1}{A} + \frac{5}{6}\,\frac{1}{n^2}\right)\sqrt{\gamma}\,\vartheta,$$

(c) from the equation for the gas as a whole,

$$\nabla p_1 = \nabla p_2 = \tfrac{1}{2}\nabla p = \tfrac{1}{2}\rho\mathbf{G},$$

and from the equation for the electrons,

$$\nabla p_2 + Ne\mathbf{E} = -(3/2\xi)N\kappa\nabla T.$$

Hence the same pressure gradient and the same electric field appear as in the previous case.

Finally, application of Eq. (5–32) to the heavy ions gives

$$\nabla p_3 - N_3 A m_1 \mathbf{G} - N_3 ne\mathbf{E} =$$

$$= K_{13}z\left(\frac{\mathbf{r}_3}{A+1} - \frac{A\mathbf{r}_1}{A+1}\right) + K_{23}z\left(\frac{\gamma\mathbf{r}_3}{A} - \mathbf{r}_2\right) \cong$$

$$\cong -n^2\,\frac{N_3}{N}\,Kz\left(\frac{\mathbf{r}_1}{\sqrt{\gamma}} + \mathbf{r}_2\right),$$

when $A + 1$ is replaced by A. Substitution of the values of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{E}$ leads to

$$\frac{\nabla N_3}{N_3} = \left[n^2\left(\frac{15}{4\sqrt{2}} + \frac{3}{2\xi}\right) - \frac{3n}{2\xi} - 1\right]\frac{\nabla T}{T} + (A - \tfrac{1}{2}n)\,\frac{m_1\mathbf{G}}{\kappa T}. \qquad (5\text{–}82a)$$

Of the terms on the right side, the first one with the factor n^2 is the most important. The numerical coefficient associated with it is approximately $2.65 + 0.80 = 3.45$. Hence we obtain

$$N_3 \qquad \text{proportional to} \qquad T^{3.45\,n^2}. \qquad (5\text{–}82b)$$

The result indicates that there is an astonishingly powerful thermal diffusion in a plasma containing heavily ionized constituents, driving these towards high-temperature regions [35].

35. See S. CHAPMAN, Thermal diffusion in ionized gases, *Proc. Phys. Soc. (London)* **72**, 353–362 (1958). There is a difference between the coefficients obtained in Eq. (5–82a) of our text and those in Chapman's Equation (33), which would lead to a coefficient $15/4\sqrt{2} = 2.65$ with n^2, instead of the somewhat larger value given here. It may be that the difference is due to the circumstance that Chapman uses the equation for binary diffusion, which will be less appropriate for the system with three kinds of particles.

The most direct approach to the problem seems to be as follows. We write Eqs. (5–32) (with the omission of irrelevant terms and with $\rho_e = 0$) in the form

$$\nabla p_s - \frac{\rho_s}{\rho}\,\nabla p - N_s e_s \mathbf{E} = \sum_t \omega_{st}\mathbf{w}_t + \sum_t \Omega_{st}\mathbf{r}_t \qquad (*)$$

and similarly Eqs. (5–35), after division by $\kappa T/m_s$:

$$\tfrac{5}{2}N_s\kappa\nabla T = \sum_t q_{st}\mathbf{w}_t + \sum_t Q_{st}\mathbf{r}_t. \qquad (**)$$

The coefficients $\omega_{st} \ldots Q_{st}$ can be obtained from the expressions for the right sides of Eqs. (5–32), (5–35). Let Q be the determinant of the Q_{st} and Q^*_{st} its minors, so that $\sum_s Q^*_{sk}Q_{st} = \delta_{kt}Q$. We solve (**) for the $\mathbf{r}_t$ and substitute the results into (*). This gives

$$\nabla p_s - \frac{\rho_s}{\rho}\,\nabla p - N_s e_s \mathbf{E} = \sum[\omega_{st} - \Omega_{sk}q_{ht}Q^*_{hk}/Q]\mathbf{w}_t + \tfrac{5}{2}\kappa\nabla T\sum\Omega_{sk}(Q^*_{hk}/Q)N_h.$$

In the case of equilibrium all $\mathbf{w}_t$ are zero. The equation then at once gives the relation between ∇p_s and ∇T, and from there, by writing $p_s = N_s\kappa T$, the relation between ∇N_s and ∇T. The equation should be applied to the case where the subscript s corresponds to the heavy ions.

In evaluating the coefficients of the equations, the point which requires attention concerns the orders of magnitude of the various terms and the decision as to what terms can be rejected. However, for a system with three kinds of particles, the minors Q^*_{st} reduce to two by two determinants, so that the work is not too complicated. The calculation according to this scheme confirmed the result given in Eq. (5–82a).

It is difficult to say whether this effect will tend to make the heavy ions predominant in the hot parts of the solar corona, since there is turbulence which may upset the equilibrium distribution assumed here.

It is possible that these thermal diffusion phenomena may be of interest in connection with nuclear reaction experiments in such apparatus as "Zeta," where a high-temperature plasma is confined by a magnetic field. If some heavy, highly charged ions appear in the plasma, it may be that they will be driven to the region of highest temperature. This point was discussed by Rosenbluth and Post with Chapman. Post considered the confinement of a plasma by a magnetic field in a long tube, where according to Spitzer a radial electrostatic field will appear. Highly ionized heavy ions might then perhaps be concentrated toward the axis. Thompson added that such a radial electric field could be responsible for producing large motions in impurity ions and consequent doppler shifts of their spectral lines.

5–18 Comments on the validity of the Boltzmann equation. Its relation to the two-particle distribution function and to the Fokker-Planck equation. Extensive discussions took place at the Symposium concerning the validity of the Boltzmann equation, in which Gross, Grad, and to a lesser extent Krook were the main participants. The discussion was followed by a deduction of the Fokker-Planck equation by Chan-Mou Tchen. The discussion was difficult to record, but Gross and Grad submitted much of their remarks in writing after the Symposium. The greater part of these written comments has been incorporated in the present section. It will be followed by some observations concerning the application of the two-particle distribution function, mainly taken from other sources; after this Tchen's contribution will be considered.

Gross [36] pointed out the necessity for a deeper analysis of the collision phenomena and of the correlation between particles in view of the following points:

(a) The thermodynamic properties of a plasma—equation of state, internal energy function, entropy function—require a knowledge of the two-particle distribution function.

(b) The Debye cloud and its behavior at high frequencies and high electric fields falls into the same category. There are observable effects, such as the high frequency and high field conductivity of plasmas, which should exhibit corrections arising from the failure of the cloud to form completely.

(c) The Holtsmark type of analysis of the line broadening problem

36. See also Gross' contribution to Section 5–8.

implicitly works on the level of multiparticle distribution function theory [37].

Grad started by observing that the divergences which arise from long-range coulomb forces make the problem of justifying or even discovering the correct form to be taken by a Boltzmann equation a very difficult one. Grad proposed to proceed at first formally, leaving the discussion until later. One should start by considering various representative lengths. Let σ be the "molecular diameter" or mean distance of closest approach

$$e^2/\kappa T,$$

and introduce the mean interparticle distance

$$\lambda_0 = N^{-1/3}$$

[not to be confused with the parameter λ defined by Eq. (5–63a)], and further the Debye length, defined by

$$d^2 = \kappa T/Ne^2,$$

and the mean free path

$$L = L'/\ln \Lambda,$$

where $L' = 1/N\sigma^2$ and $\Lambda = d/\sigma$. Numerical factors have been suppressed [in this sense the Λ defined here is the same quantity as the one given before by Eq. (5–21)]. In rough comparisons the distinction between L and L' may be ignored.

In a fully ionized plasma we have

$$\sigma \ll \lambda_0 \ll d \ll L';$$

more precisely, the relative orders of magnitude can be seen from

$$\sigma\, d^2 = \lambda_0^3 = \sigma^2 L', \qquad d^2 = \sigma L'. \tag{*}$$

The Debye length d measures the effective range of two-particle correlations. Forces due to particles separated by more than this distance are statistically independent and can be taken care of by the averaged charge and current source terms in Maxwell's equations. Forces arising from particles closer than the mean interparticle distance λ_0 can be handled by the binary collision analysis used in obtaining the conventional Boltzmann equation. There is left the important range of distances between

37. The problem of line broadening is of importance in spectroscopic research concerning the state of the gas at high temperatures. It has been considered by Post in Section 2–4.

λ_0 and d in which we find particles which are correlated and yet cannot be split into pairs: the Debye sphere contains many particles, since $Nd^3 = (d/\lambda_0)^3 \gg 1$. To be precise, binary collision analysis can be used up to a distance $\hat{\sigma}$, where $\sigma \ll \hat{\sigma} \ll \lambda_0$. The omitted range between $\hat{\sigma}$ and d is characterized by the superposition of many *grazing* deflections. Assuming that these impulses are random and independent, we have a Markoff process and, following Chandrasekhar and Spitzer, we conclude that the resultant contribution to the Boltzmann equation is a second-order differential operator (Fokker-Planck term) in addition to the integral operator (Boltzmann term) which arises from the large deflection binary collisions. If we examine the conventional Boltzmann collision term in the range beyond $\hat{\sigma}$ (where it is, presumably, invalid) we find that, on making a grazing collision approximation, it yields exactly the same Fokker-Planck term as the one just described [38]. This can be seen *a priori*. The original description was of a stochastic process with many *simultaneous* independent small deflections. The Boltzmann description corresponds to a hypothetical physical situation in which there is a sequence in time of many independent random *binary* deflections. Although the two physical pictures are radically different, the mathematical models are clearly the same. The simplest procedure is then to include in the Boltzmann term grazing deflections out to the distance d, instead of taking a sum of Boltzmann and Fokker-Planck terms.

It is true that the justification that has been given of the use of a Fokker-Planck equation is incomplete. However, any analysis which demonstrates the validity of a Fokker-Planck equation also justifies (in this roundabout way) the validity of the binary collision Boltzmann equation.

It is necessary to introduce a cutoff at the distance d to prevent divergent integrals. To avoid this *ad hoc* procedure one should use the correct shielded potential. This is

$$\exp(-r/d)/r$$

rather than the $1/r$ in thermodynamic equilibrium, but it must be recomputed for more general use. This, however, is a crucial point. The shielded potential is roughly equivalent to the two-particle or radial distribution function [39]. The basic question is how to approximate it in terms of only the one-particle distribution.

38. H. GRAD, Thermonuclear reaction rates in an electrical discharge, Inst. Math. Sci., NYU, NYO-7977 (January 1958).

39. What one must actually compute is the two-particle correlation function at two different times. For example, the radial distribution function in equilibrium is $\exp(-r/d)/r$ even for moving particles. This single-time function, however, is not the correct shielding distribution to be inserted in the collision term of the Boltzmann equation.

There exist several examples in statistical mechanics where a lower order level of information suffices to determine the complete state of a system in the limit as $N \to \infty$. In equilibrium the complete state is determined by the thermodynamic variables alone. There are several possibilities in the nonequilibrium case. If we take $N\sigma^2$ fixed and let $N\sigma^3 \to 0$, we obtain, in the limit, a perfect gas which satisfies the binary collision Boltzmann equation [40]; the mean free path is fixed in this limit. In this limit, the two-particle distribution (which must be found to compute the evolution in time of the one-particle distribution) can be shown to become dependent upon the one-particle distribution. On the other hand, if we take $N\sigma^3$ fixed and let $N\sigma^2 \to \infty$, we obtain the Euler equations of fluid motion in the limit [41]. In this case the reduction is more radical; the local thermodynamic state is a complete description. From a careful study of these examples it becomes clear that one cannot hope to find a Boltzmann-like description (in terms of a one-particle distribution) for a dense gas or liquid, but only for a rarefied gas [42].

In the case of a plasma, the situation is somewhat like that in a liquid; many-body interactions are important. However, this is a very special type of "liquid," since the multiple interactions are all weak. To obtain any strictly mathematical simplification over that of an n-body problem, we must take some kind of limit. When it is desired to keep the effect of collisions finite (L is finite) and to make $\sigma \to 0$, it is seen from equation (*) above that the mean interparticle distance λ_0 and the Debye length d must also approach zero. In this limit all that is left of collisions is the Fokker-Planck term, since the binary collision term is smaller by the factor $\ln \Lambda = \ln (d/\sigma)$. On the other hand, it may very well be that keeping terms "to the next order," i.e. to the order $1/\ln \Lambda$ is permissible, just as dense gas corrections may possibly be kept in the usual Boltzmann equation [42] and just as Navier-Stokes terms arise in the fluid limit, $N\sigma^3 =$ constant, when higher order terms are kept. Thus one can expect that the Boltzmann equation, including grazing collisions out to a distance d, may be correct.

The problem that remains is to actually determine the two-particle distribution in the limit of large d/σ. This we can do by using the Boltzmann equation itself. It is a well-defined problem (but, as yet, solved only

40. See H. GRAD, Principles of the kinetic theory of gases, *Handbuch der Physik* **12** (1958).

41. C. B. MORREY, On the derivation of the equations of hydrodynamics from statistical mechanics, *Comm. on Pure and Applied Math.* **8**, 279–326 (1955).

42. It is shown in the article mentioned in Ref. 40 (Section 14) that a hypothetical virial-type expansion, such as the one introduced by Bogoliubov, expressing the two-particle distribution in terms of the one-particle distribution, can be valid, at best, to first order in the density.

for special cases [43]) to find the flow of a plasma past an external (macroscopic) fixed charge. This solution should then be used in the collision term of the Boltzmann equation to replace the *ad hoc* cutoff at the distance d. First of all, it should be noted that $L \gg d$ if $d \gg \sigma$; consequently, the collision term in the Boltzmann equation (which is not precisely known until we have solved the shielding problem) is largely irrelevant to the solution of this flow problem past a charge, since this flow has dimensions of the order of d. Secondly, we note that the field due to an electron or ion can be considered to be macroscopic in the solution of this flow problem, since its effective range is of the order of d, which is much larger than σ. This situation can be compared to that in the ordinary virial expansion of the radial distribution function. The leading term, $\exp(-\phi/\kappa T)$, is exact for a potential ϕ which is macroscopic in range. Only for a potential which is microscopic in range must the formula be corrected [44].

The procedure described here will allow a correct Boltzmann equation to be computed. However, the conventional expedient of adopting a cutoff at d instead of a shielding potential is probably quite good [45, 46].

Finally, the following observations were made by Krook, mainly with reference to the material of Sections 5–19 and 5–20. His contention is that the distribution functions, of any order 1, 2, . . . , do not describe the state of a real physical system at all. Omitting even the ticklish question of

43. Unpublished work of E. RUBIN and S. RAND. Also see L. KRAUS and K. M. WATSON, Plasma motions induced by satellites in the ionosphere, *The Physics of Fluids*, **1**, 480–488, 1958, where the same mathematical problem is analyzed in a different physical situation. This "macroscopic" procedure is much simpler than the collective coordinate approach of BOHM and PINES (see Section 5–23 here).

44. For a particle at rest this very formula can be used to obtain the spherically symmetric shielding potential, $\exp(-r/d)/r$.

45. R. LIBOFF, Computation of the transport coefficients using the shielded coulomb potential, Inst. Math. Sci., NYU, NYO-8669.

46. GRAD in his letter added the remark that some authors, among them Krook and Gross, disagreed with his claim that it is possible to obtain a determined equation for the one-particle distribution in a plasma, rather than to have to rely on higher correlations.

The main point of Grad's argument is to show that the two-particle distribution can be satisfactorily approximated in terms of the one-particle distribution. More basically, the question is whether there exists any equation at all of the Boltzmann or Fokker-Planck type for the one-particle distribution. This possibility is usually assumed rather than demonstrated. For example, BOGOLIUBOV and H. S. GREEN make this assumption for dense gases, a case for which it is not even true. For a critical discussion of this and related points, Grad referred to his article in *Handbuch der Physik* **12**, which has just appeared.

symmetries with respect to like particles in the classical formalism, the distribution functions can be defined rigorously only as specifying the state of an ensemble of systems. The Liouville equation is an equation of motion for the ensemble and not for the system of interest. In equilibrium statistical mechanics, where one inquires only into the average values of certain local physical variables, a reasonable identification can be made between properties of the ensemble and properties of the system of interest. In kinetic theory, on the other hand, the situation is much more complex.

The term on the right side of Eq. (5–83), given in Section 5–19 below, involves an average force calculated with the two-particle distribution function as weight factor. However, to derive the Boltzmann form of the interaction term (compare H. S. Green) we must change our point of view for calculating the average force arising from interaction of particles within the correlation sphere. This may indeed be a very drastic step, so that the Boltzmann equation, which already does not describe the behavior of a real physical system, does not even describe the behavior of an ensemble of systems. Krook thinks that nevertheless a case can be made for the view that the kinetic equations do represent the state and behavior of the physical system for a limited time, the description of the behavior becoming continuously worse with increasing time.

The Bogoliubov-Kirkwood type of cutoff is on somewhat safer ground, but still only represents in an approximate way the behavior of the ensemble and not of the physical system.

If we consider the actual physical problems which kinetic theory is designed to handle, further difficulties of principle appear. Systems are prepared to be in particular initial states by means of macroscopic handling operations. The only properties that we can specify are then the values of certain macroscopic fields; the number of such macroscopic variables may of course be as large as we please, in principle, but the number must still be finite. We have (1) to translate this initial macroscopic data into a specific set of initial distribution functions, (2) to solve the corresponding initial value problem for the kinetic equations, and (3) to translate the microscopic solution back into terms of macroscopic variables. Step (1) is by no means clear-cut. We have already remarked on the difficulties associated with step (2). The conceptual difficulties become very much clearer if we ask the questions in a somewhat different way, in particular, if we inquire into the *correlation* between the state of the system at time t_1 and time $t_2 > t_1$. Krook hopes to deal with these questions more fully in a forthcoming paper. In his opinion the formalism of kinetic theory can only provide a description that becomes progressively more inexact. The problem of determining a characteristic time for this decrease of information would be a tough one.

5–19 Application of the two-particle distribution function. Relation to Boltzmann's expression for binary collisions. After these general considerations and in order to have a bridge towards Chan-Mou Tchen's deduction of the Fokker-Planck equation, it may be useful to give some attention to the fundamental equation which connects the distribution function for a single particle with the two-particle function. This equation is the first one of a set of successive equations, linking the distribution functions for increasing numbers of particles, which have been derived by such various authors as Born and Green, Kirkwood, and Bogoliubov by an integration procedure applied to the so-called Liouville equation [47]. The first equation of this set is

$$\frac{\partial F_s}{\partial t} + \xi_{sh}\frac{\partial F_s}{\partial x_{sh}} + \frac{f_{sh}}{m_s}\frac{\partial F_s}{\partial \xi_{sh}} = \frac{1}{m_s}\sum_t N_t \iint d\mathbf{x}_t\, d\xi_t \frac{\partial \Phi_{st}}{\partial x_{sh}}\frac{\partial F_{st}}{\partial \xi_{sh}}. \tag{5–83}$$

In this equation the function F_{st} is the two-particle distribution function, depending upon the variables ξ_{sh}, ξ_{th}, x_{sh}, x_{th} (it is necessary to label the coordinates with subscripts to indicate the particles to which they refer), and the time. The function Φ_{st} is the potential function for the interaction between the particles s and t (coulomb interaction for charged particles), and f_{sh} as before represents the exterior forces, for instance the forces e_sE_h derived from an electric field imposed from the outside. A magnetic field will not be introduced. The functions F_s and F_{st} in (5–83) are normalized in such a way that

$$\int F_s\, d\xi_s = 1, \qquad \iiint F_{st}\, d\mathbf{x}_t\, d\xi_s\, d\xi_t = V,$$

where V represents the total volume of the gas or of the plasma over which the coordinate integration is extended; this volume is supposed to be very large compared with the Debye sphere. (The normalization used for F_s is different from that used before.)

47. See: M. Born and H. S. Green, A general kinetic theory of liquids I. The molecular distribution functions, *Proc. Roy. Soc. (London) A* **188,** 10–18 (1946); H. S. Green, *Molecular Theory of Fluids* (Amsterdam, 1952), p. 128, Eq. (2.7).

J. G. Kirkwood, The Statistical mechanical theory of transport processes I. General theory, *J. Chem. Phys.* **14,** 180–201 (1946); II. Transport in Gases, *ibid.* **15,** 72–76 (1947).

N. Bogoliubov, Kinetic equations, *J. of Phys. (U.S.S.R.)* **10,** 265–274 (1946); *Problems of a Dynamical Theory in Statistical Physics,* State Techn. Press, Moscow, 1946 (transl. by Geoph. Res. Directorate, AFCRC, Jan. 1959).

G. E. Uhlenbeck, The Boltzmann equation, in Seminar on Applied Mathematics, Boulder, Colorado, 1957 (to be published by the Am. Math. Soc.).

In carrying out the integration with respect to $d\mathbf{x}_t$ attention must be given to the various distances considered by Grad. We shall briefly sketch how the standard formula (5–2) of the right side of the Boltzmann equation is related to the integral of the right side of (5–83) over the first domain, which extends up to the distance σ. We follow a method indicated by Green [48].

Within this domain it can be assumed that terms due to three-particle distribution functions F_{str} may be left out of account. It is then possible to make use of an approximate form of the equation which is satisfied by F_{st}. The full equation will be given in the next section, Eq. (5–87). When we leave out the terms with the function F_{str} and at the same time assume that the exterior forces f_{sh} have no appreciable influence upon F_{st} within this region, the equation takes the form

$$\frac{\partial F_{st}}{\partial t} + \xi_{sh}\frac{\partial F_{st}}{\partial x_{sh}} + \xi_{th}\frac{\partial F_{st}}{\partial x_{th}} = \frac{1}{m_s}\frac{\partial \Phi_{st}}{\partial x_{sh}}\frac{\partial F_{st}}{\partial \xi_{sh}} + \frac{1}{m_t}\frac{\partial \Phi_{st}}{\partial x_{th}}\frac{\partial F_{st}}{\partial \xi_{th}}. \tag{5–84}$$

With the aid of this equation we obtain the following expression for the right side of (5–83):

$$-\frac{1}{m_t}\sum_t N_t \iint d\mathbf{x}_t\, d\xi_t \frac{\partial \Phi_{st}}{\partial x_{th}}\frac{\partial F_{st}}{\partial \xi_{th}} + \sum_t N_t \iint d\mathbf{x}_t\, d\xi_t \left(\frac{\partial F_{st}}{\partial t} + \xi_{sh}\frac{\partial F_{st}}{\partial x_{sh}} + \xi_{th}\frac{\partial F_{st}}{\partial x_{th}}\right).$$

The first term of this expression vanishes when the integration with respect to $d\xi_t$ is performed. In order to transform the other term, we write

$$\mathbf{x}_t - \mathbf{x}_s = \mathbf{r}, \qquad \xi_t - \xi_s = \mathbf{g}.$$

Since the location and the velocity of the particle s are fixed, we can use $\mathbf{r}$ and $\mathbf{g}$ as integration variables, and the remaining part of the expression can be written

$$\sum_t N_t \int d\mathbf{g} \int d\mathbf{r} \left(\frac{\partial F_{st}}{\partial t} + \xi_{sh}\frac{\partial F_{st}}{\partial x_{sh}} + g_h\frac{\partial F_{st}}{\partial r_h}\right). \tag{5–85a}$$

The most important term in this formula is

$$\sum_t N_t \int d\mathbf{g} \int d\mathbf{r}\, g_h \frac{\partial F_{st}}{\partial r_h}, \tag{5–85b}$$

48. H. S. Green, *Molecular Theory of Fluids* (Amsterdam, 1952), pp. 218–225 (derivation of Boltzmann's equation, and corrections to Boltzmann's equation).

since the derivative of F_{st} with respect to the variable $\mathbf{r}$ will be large within our domain, while the derivatives with respect to $\mathbf{x}_s$ for fixed $\mathbf{r}$, and with respect to the time, will be of moderate magnitude.

We take the integration with respect to $d\mathbf{r}$ first and assume that the region over which this integration is extended is enclosed by a surface Σ, surrounding the particle s to the extent indicated before. In this integration the vector $\mathbf{g}$ has a fixed direction. From every element $d\Sigma$ of Σ we draw a normal $\mathbf{n}$ outward from $d\Sigma$ and we call (n, g) the angle between $\mathbf{n}$ and $\mathbf{g}$. The surface Σ can be divided into two parts, on one of which $\cos(n, g)$ is positive, while on the other $\cos(n, g)$ is negative. Performing the integration with respect to $d\mathbf{r}$, we obtain

$$\int d\mathbf{r}\, g_h \frac{\partial F_{st}}{\partial r_h} = g \int (F_{st}) \cos(n, g)\, d\Sigma,$$

where (F_{st}) is the value of the function F_{st} at the element $d\Sigma$, at the time t and for the location x_s for which the integration is performed. Consider the part of Σ for which $\cos(n, g)$ is *negative;* we then can write

$$-\cos(n, g)\, d\Sigma = dS,$$

where dS is an element of the cross section for collisions with relative velocity $\mathbf{g}$, equivalent to the product $b\, db\, d\epsilon$ in our previous expressions. Hence when we consider these parts of the integral alone and carry out the integration with respect to $d\mathbf{g}$, we obtain the result

$$-\iint (F_{st}) g\, dS\, d\mathbf{g},$$

referring to the particles t entering the surface Σ or, as we can say, entering into a collision process with the particle s. On the other hand, when we consider the part of Σ for which $\cos(n, g)$ is *positive* and carry out the integration with respect to $d\mathbf{g}$, we obtain the result

$$+\iint (F_{st})' g\, dS\, d\mathbf{g},$$

referring to the particles t leaving the surface Σ, that is, coming out of a collision with the particle s. Thus the result of the integration can be written

$$\iint (F_{st})'\, g\, dS\, d\mathbf{g} - \iint (F_{st})\, g\, dS\, d\mathbf{g}, \tag{5–86}$$

corresponding to the two terms in the standard formula (5–2) for binary collisions. There is the difference that in (5–2) we considered the particles t and s as completely independent before they entered into a collision,

whereas here the use of the function F_{st} instead of the product $F_s F_t$ allows for the existence of a correlation produced by the influence of the particles in the region surrounding our present domain of integration.

There remain the terms with derivatives with respect to the time and to x_{sh} for fixed $\mathbf{r}$ in (5–85). These terms introduce corrections related to changes in the function F_{st} occurring during the transit of the particle t through the space enclosed by Σ, that is, changes of F_{st} occurring within the duration of a single collision. We may expect these corrections to be small, although they can be of some importance with nonuniform fields or with changes of the field occurring in time. Attention to these corrections is given by Green in reference [48], but we shall not pursue this point here.

5–20 Reduction of the right side of Eq. (5–83) to an expression of the Fokker-Planck type. We shall now consider the integral with respect to $d\mathbf{x}_t$ occurring in (5–83) over the region of grazing deflections. The exposition in the following lines is patterned mainly after Tchen's contribution to the discussion [49].

For this purpose we need the first two equations of the set mentioned at the beginning of the previous section, that is, Eq. (5–83) given in that section, and the following equation for the function F_{st}:

$$\frac{\partial F_{st}}{\partial t} + \xi_{sh} \frac{\partial F_{st}}{\partial x_{sh}} + \xi_{th} \frac{\partial F_{st}}{\partial x_{th}} + \frac{f_{sh}}{m_s} \frac{\partial F_{st}}{\partial \xi_{sh}} + \frac{f_{th}}{m_t} \frac{\partial F_{st}}{\partial \xi_{th}} =$$

$$= \frac{1}{m_s} \frac{\partial \Phi_{st}}{\partial x_{sh}} \frac{\partial F_{st}}{\partial \xi_{sh}} + \frac{1}{m_t} \frac{\partial \Phi_{st}}{\partial x_{th}} \frac{\partial F_{st}}{\partial \xi_{th}} +$$

$$+ \sum_r N_r \iint d\mathbf{x}_r\, d\xi_r \left(\frac{1}{m_s} \frac{\partial \Phi_{sr}}{\partial x_{sh}} \frac{\partial F_{str}}{\partial \xi_{sh}} + \frac{1}{m_t} \frac{\partial \Phi_{tr}}{\partial x_{th}} \frac{\partial F_{str}}{\partial \xi_{th}} \right). \tag{5–87}$$

The normalization used here is similar to that introduced in connection with Eq. (5–83). The coordinate dependence of the functions F_{st} and F_{str} is mainly through the differences $\mathbf{x}_s - \mathbf{x}_t$, $\mathbf{x}_s - \mathbf{x}_r$, etc., although the coordinates themselves will enter if the state of the gas is not uniform

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49. C. M. Tchen, Kinetic equations for a plasma with unsteady correlations, *Phys. Rev.* **114**, 394–411 (1959). Tchen's treatment in part follows a paper by S. W. Temko, On the deduction of the Fokker-Planck equation for a plasma (Russian text), *Zhur. Eksptl. i Teort. Fiz.*, **31**, 1021–1026 (1956); Am. Inst. Phys. translation in *Soviet Physics JETP* **4**, 898–903 (1957). Tchen's paper clears up several points which were left rather vague by Temko.

in space. In several papers the momenta $p_{sh} = m_s \xi_{sh}$, etc. are used as variables in place of the velocities ξ_{sh}; then q_{sh} is written for x_{sh}, etc.

The treatment of the equations is still in a stage of development. It is usually assumed that the effect of the exterior forces upon the two- and more particle distribution functions may be neglected; we shall drop the terms containing these forces from (5–87). In that case, we can consider F_{st} and F_{str} to be functions of the coordinate differences, with the coordinates of the particle s as a slowly varying parameter.

The main problem is how to cope with the three-particle function F_{str}. Tchen made the suggestion that it be approximated by the expression

$$F_{str} \cong F_s F_{tr} + F_t F_{sr} + F_r F_{st} - 2F_s F_t F_r. \tag{5–88a}$$

When we write

$$F_{st} = F_s F_t + F'_{st},$$

which is always possible, Eq. (5–88a) is equivalent, with

$$F_{str} = F_s F_t F_r + F_s F'_{tr} + F_t F'_{sr} + F_r F'_{st}. \tag{5–88b}$$

Equations (5–83) and (5–87) can then be transformed into two equations containing only functions of type F_s and F'_{st}. Certain terms drop out in the integrations, and the results are

$$\frac{\partial F_s}{\partial t} + \xi_{sh} \frac{\partial F_s}{\partial x_{sh}} + \frac{f_{sh}}{m_s} \frac{\partial F_s}{\partial \xi_{sh}} = \frac{1}{m_s} \sum_t N_t \iint d\mathbf{x}_t \, d\xi_t \, \frac{\partial \Phi_{st}}{\partial x_{sh}} \frac{\partial F'_{st}}{\partial \xi_{sh}}, \tag{5–89}$$

$$\begin{aligned} \frac{\partial F'_{st}}{\partial t} + g_h \frac{\partial F'_{st}}{\partial x_{sh}} = {} & \frac{\partial \Phi_{st}}{\partial x_{sh}} \left(\frac{F_t}{m_s} \frac{\partial F_s}{\partial \xi_{sh}} - \frac{F_s}{m_t} \frac{\partial F_t}{\partial \xi_{th}} \right) + \\ & + \frac{\partial \Phi_{st}}{\partial x_{sh}} \left(\frac{1}{m_s} \frac{\partial F'_{st}}{\partial \xi_{sh}} - \frac{1}{m_t} \frac{\partial F'_{st}}{\partial \xi_{th}} \right) + \\ & + \sum_r N_r \iint d\mathbf{x}_r \, d\xi_r \left(\frac{\partial \Phi_{sr}}{\partial x_{sh}} \frac{F'_{tr}}{m_s} \frac{\partial F_s}{\partial \xi_{sh}} + \frac{\partial \Phi_{tr}}{\partial x_{th}} \frac{F'_{sr}}{m_t} \frac{\partial F_t}{\partial \xi_{th}} \right). \end{aligned} \tag{5–90}$$

We thus have arrived at a closed system of equations. In the case of a simple plasma there will be two single-particle functions and three two-particle functions; also there are two equations of type (5–89) and three equations of type (5–90). The equations are nonlinear and require the introduction of various approximations, which cannot yet be made fully satisfactory. Here we can indicate only a bare outline of the method of treatment and refer the reader to reference [49].

When provisionally the right side of (5–90) is considered as a given quantity, the equation can be formally integrated along the characteristics of the differential operator on the left side. Since in practically all cases the distance between two particles can be considered as infinite for $t = -\infty$, we integrate from $t = -\infty$ to the current value of t with initial condition $F'_{st}(-\infty) = 0$.

Fourier transforms are introduced to describe the dependence of the functions F'_{st} upon the coordinate differences $\mathbf{x}_s - \mathbf{x}_t$. The dependence upon the velocity components is not subjected to any transformation. We write

$$\Phi_{st} = e_s e_t \int d\boldsymbol{\nu} \, \frac{1}{2\pi^2 \nu^2} \exp\left[i\nu_h (x_{sh} - x_{th}) \right] \tag{5–91}$$

and

$$F'_{st} = e_s e_t \int d\boldsymbol{\nu} \, \varphi_{st}(\boldsymbol{\nu}) \exp\left[i\nu_h (x_{sh} - x_{th}) \right]. \tag{5–92}$$

It is necessary to observe that the application of Fourier integrals makes it less easy to take account of the limits of the domain we are considering, both on the interior side and on the exterior side. With regard to the latter point, it will appear later that we can arrive at integrals converging for $\nu = 0$, which means that the results obtained are independent of phenomena happening at large distances. The form of the integrals in the neighborhood of $\nu = 0$ will automatically lead to expressions containing the Debye screening effect. It is different with the behavior of the integrals for large values of ν. Here we shall make use of the following artifice: what actually happens to the function F_{st} on the occasion of close encounters between the particles s and t must be treated by a method of the type indicated in the preceding section and will lead to the appearance of a binary collision term to be applied for close encounters only. In treating the cooperative effect of the wider surroundings, we shall therefore exclude the effects which appear at close encounters; we can do this by assuming that the potential function Φ takes a *constant* value in the interior domain, so that no forces will be operating there. When the coulomb potential function has been altered in the appropriate way, we extend the integration over the full space and can make use of the Fourier expressions without difficulty.

To assure that Φ_{st} shall take a constant value within a domain of radius a, we replace the fraction $1/2\pi^2\nu^2$ occurring in (5–91) by $\sin \nu a / 2\pi^2 \nu^3 a$. Instead of working with this somewhat complicated formula, we can just as well keep to Eq. (5–91), provided we cut off the integration with respect to ν at a maximum value of the order $1/a$. This gives a slightly different behavior of the potential in the interior domain, but again there is a

finite maximum value and to the order of accuracy that can be obtained in this type of calculation the result is quite sufficient. Since it has been mentioned before that the radius of the interior domain should be of the order $e^2/\kappa T$, we use a cutoff value for ν equal to $\kappa T/e^2$ [50].

Equation (5–90) can now be transformed into an equation for the function $\varphi_{st}(\nu)$. It is convenient to introduce an integration with respect to $d\xi_t$ and to write

$$\int d\xi_t \, \varphi_{st}(\nu) = H_s^t(\nu).$$

Equation (5–90) then is transformed into an equation for the function H_s^t.

At the same time, the right side of Eq. (5–89) becomes transformed into

$$-\frac{(2\pi)^3 i e_s^2}{m_s} \frac{\partial}{\partial \xi_{sh}} \int d\nu \, \frac{\nu_h}{2\pi^2 \nu^2} \sum_t N_t e_t^2 H_s^t(\nu). \tag{5–93}$$

We shall not write out completely the transformed equation (5–90), but observe that on the right side it will contain three groups of terms, corresponding to the three groups of Eq. (5–90). These three groups play different parts. The first group, depending directly upon the functions F_s and F_t, is the most typical. When suitable approximations are introduced, it leads to an expression of the Fokker-Planck type and thus fulfills the first objective of the calculation. It appears, however, that some of the necessary integrations with respect to $d\nu$ become divergent for $\nu \to 0$ (corresponding to large distances in physical space) and for $\nu \to \infty$ (corresponding to very small distances in physical space). An approximation for the third group of terms coming from Eq. (5–90) (the term with the double integral over the coordinates and the velocity components of the third particle r) introduces a correction term, which ensures the convergence of the integration for $\nu = 0$. This correction at the same time introduces the Debye screening effect, ensuring a finite value for the effective collision cross section. Finally, the second group of terms introduces a correction which is of some importance at the larger values of ν, although it may vanish when all particles have unit charges, so that there are as many positive as there are negative charges in the plasma. The correction

50. The author found that several features of Eqs. (5–83) and (5–87) can be investigated almost equally well by integration in ordinary coordinate space, instead of introducing Fourier transforms. Such integration in particular makes it easier to separate the contributions of close encounters from the grazing deflections which are taken together in the Fourier integrals. A short account has been given by J. M. Burgers, Some aspects of particle interaction in gases, Inst. for Fluid Dynamics and Applied Mathematics, U. of Maryland, Technical Note BN-176, pp. 13–21 (June 1959).

does not materially influence the results when we introduce a cutoff for the integration with respect to ν at a limit of the order $\kappa T/e^2$, as was proposed before.

5–21 An expression of the Fokker-Planck type. (Cont.) When we restrict to the terms deriving from the first group on the right side of Eq. (5–90) the transformed equation has the form

$$(H_s^t)_{\mathrm{I}} = \frac{i\nu_h}{2\pi^2\nu^2} \int_0^\infty d\tau \int d\xi_t \left[\frac{F_t}{m_s} \frac{\partial F_s}{\partial \xi_{sh}} - \frac{F_s}{m_t} \frac{\partial F_t}{\partial \xi_{th}} \right]_{t-\tau} \exp\,(i\nu_h g_h \tau). \tag{5–94}$$

Here τ is an auxiliary variable arising from the integration with respect to the time performed upon (5–90) along the characteristics. The subscript $t - \tau$ after the expression within [ ] indicates that its value must not be taken for the instant t, but for the instant $t - \tau$. However, we shall suppose that the quantity within the [ ] varies very slowly with t and that the combined effect of the integrations with respect to $d\tau$ and to $d\xi_t$ will ensure a sufficiently rapid convergence to prevent that a serious error would arise from this approximation [51].

As another approximation, we shall replace the unknown function F_t in the [ ] expression by a simple Maxwellian expression of the type indicated in formula (5–13), with $c_{sh} = \xi_{sh} - u_{sh}$, so as to allow for different flow velocities of the constituents of the plasma. For simplicity, we assume that there is only one temperature T.

These approximations reduce $(H_s^t)_{\mathrm{I}}$ to a linear function of F_s and of its first derivatives with respect to the velocity components ξ_{sh}. When this result is substituted into formula (5–93) an expression of the Fokker-Planck type is obtained, which can be written

$$-\frac{\partial}{\partial \xi_{sh}} \{(\overline{A}_h)_{\mathrm{I}} F_s\} + \frac{\partial}{\partial \xi_{sh}} \left\{ (\overline{B}_{hk})_{\mathrm{I}} \frac{\partial F_s}{\partial \xi_{sk}} \right\}, \tag{5–95}$$

or, alternatively,

$$-\frac{\partial}{\partial \xi_{sh}} \{(A_h)_{\mathrm{I}} F_s\} + \frac{\partial^2}{\partial \xi_{sh} \partial \xi_{sk}} \{(B_{hk})_{\mathrm{I}} F_s\}, \tag{5–96a}$$

51. When it is desired to take account of the subscript $t - \tau$ added to the bracketed expression in Eq. (5–94), use can be made of a series development of this expression with respect to powers of τ. The further reduction will then lead to a more complicated result than the usual Fokker-Planck formula, in which successive derivatives of F_s and $\partial F_s/\partial \xi_{sh}$ with respect to the time play a part. See Tchen, Ref. 49.

where

$$(A_h)_{\mathrm{I}} = (\overline{A}_h)_{\mathrm{I}} + \frac{\partial(\overline{B}_{hk})_{\mathrm{I}}}{\partial \xi_{sk}}; \qquad (B_{hk})_{\mathrm{I}} = (B_{hk})_{\mathrm{I}}. \tag{5-96b}$$

The expressions for the coefficients A and B will be given in the next section.

If we introduce Maxwellian expressions for both F_s and F_t into (5–94), both referring to the *same* mass flow velocity and to the same temperature, we obtain

$$(H_s^t)_{\mathrm{I}} = -\frac{F_s}{2\pi^2 \nu^2 \kappa T}. \tag{5-97}$$

When this is substituted into (5–93), the integrand becomes an odd function of v_h and the integral vanishes. This proves that a Maxwellian equilibrium distribution for the velocity components of all constituents of the mixture will not change in the course of time through the interactions considered here. The approximation (5–97), nevertheless, is convenient in the reduction of the terms deriving from the third group on the right side of Eq. (5–90). These terms assume the form

$$i \sum_r N_r e_r^2 \frac{4\pi v_h}{v^2} \int_0^\infty d\tau \int d\xi_t \int d\xi_r \left\{ \frac{\varphi_{tr}(-\nu)}{m_s} \frac{\partial F_s}{\partial \xi_{sh}} - \frac{\varphi_{sr}(\nu)}{m_t} \frac{\partial F_t}{\partial \xi_{th}} \right\} \exp(i v_h g_h \tau).$$

Again we neglect that the expression between { } should be taken at the time $t - \tau$. The whole expression must be added to the right side of (5–94). Since the expression has an extra integration in it, it seems possible to make use of a further approximation for its reduction. When it is assumed that we may replace

$$\varphi_{tr}(-\nu) \qquad \text{by} \qquad (F_t/F_s)\varphi_{sr}(\nu),$$

it is found that the whole expression passes into

$$(2\pi)^3 \sum_r N_r e_r^2 \frac{H_s^r}{F_s} (H_s^t)_{\mathrm{I}},$$

and when we replace $(H_s^t)_{\mathrm{I}}$ by the approximation (5–97), it becomes

$$-\frac{4\pi}{\nu^2 \kappa T} \sum_r N_r e_r^2 H_s^r.$$

We now replace (5–94) by

$$(H_s^t)_{\mathrm{II}} = (H_s^t)_{\mathrm{I}} - \frac{4\pi}{\nu^2 \kappa T} \sum_r N_r e_r^2 H_s^r. \tag{5-98}$$

We assume that the H_s^r occurring on the right side can be read as $(H_s^r)_{\mathrm{II}}$. Then it is easy to obtain the result

$$\sum_t N_t e_t^2 (H_s^t)_{\mathrm{II}} = \frac{\nu^2 r_D^2}{1 + \nu^2 r_D^2} \sum_t N_t e_t^2 (H_s^t)_{\mathrm{I}}, \tag{5-99}$$

where we have made use of the definition of r_D given in Eq. (5–20).

In view of Eq. (5–93) it follows that the expressions (5–95) and (5–96a) retain their general forms, with second approximations to the coefficients A and B obtained from the first approximations by multiplication with the factor $(\nu^2 r_D^2)/(1 + \nu^2 r_D^2)$. This at once guarantees the convergence of all integrals at the limit $\nu = 0$, as will be seen from Eqs. (5–101).

We shall not go into the reduction of the second group of terms of Eq. (5–94), but refer the reader to reference [49].

5–22 Coefficients of the Fokker-Planck expression. Mean loss of momentum. Debye potential field. We introduce the following auxiliary functions of ξ_{sh}:

$$E = \int_0^\infty d\tau \exp\left[\frac{-\nu^2 \kappa T}{2m_t} \tau^2 - i\nu_h(\xi_{sh} - u_{th})\tau \right],$$
$$E_l^* = \frac{1}{m_t} \frac{\partial E}{\partial \xi_{sl}}. \tag{5-100}$$

The second approximations to the coefficients A and B of the Fokker-Planck expression can then be written:

$$\overline{A}_h = \frac{2e_s^2 r_D^2}{\pi m_s} \int d\nu \frac{\nu_h \nu_l}{\nu^2(1 + \nu^2 r_D^2)} \sum_t N_t e_t^2 E_l^*,$$
$$\overline{B}_{hk} = \frac{2e_s^2 r_D^2}{\pi m_s^2} \int d\nu \frac{\nu_h \nu_k}{\nu^2(1 + \nu^2 r_D^2)} \sum_t N_t e_t^2 E. \tag{5-101}$$

It follows from (5–96b) in connection with (5–101) and the second line of Eq. (5–100) that

$$A_h = \frac{m_s + m_t}{m_s} \overline{A}_h. \tag{5-102}$$

When the integrals are worked out, we find

$$A_h = -\frac{4\pi e_s^2 \ln \Lambda^*}{m_s} \sum_t \frac{N_t e_t^2}{\mu_{st}} \frac{w_h}{w^3} \left\{ \operatorname{erf} \frac{w}{a_t} - \frac{2}{\sqrt{\pi}} \frac{w}{a_t} \exp\left(\frac{-w^2}{a_t^2} \right) \right\}, \tag{5-103}$$

where

$$w_h = \xi_{sh} - u_{th} \quad \text{and} \quad a_t = (2\kappa T/m_t)^{1/2}.$$

Further, $\mu_{st} = m_s m_t/(m_s + m_t)$, while Λ^* is approximately the quantity defined in Eq. (5–21), apart from a correction term connected with the second group of terms of Eq. (5–94) for which we refer the reader to reference [49].

It is well known that the quantity A_h, occurring in the form (5–96a) of the Fokker-Planck expression, measures the mean gain of momentum experienced by a particle s of given speed ξ_s in its collisions with particles t of all possible speeds. This quantity is now given by Eq. (5–103). As will be seen, its direction is opposite to that of the vector $\boldsymbol{\xi}_s - \mathbf{u}_t$, that is, the velocity vector of the particle s with respect to the mean speed of all particles t. For small values of w/a_t Eq. (5–103) reduces to an expression linear in $\mathbf{w}$, whereas for large values of w/a_t the loss of momentum becomes proportional to $\mathbf{w}/w^3$. This is similar to the result obtained in the discussion of the runaway phenomenon in Section 5–15. A complete agreement with Eq. (5–75a) is reached when we calculate the value

$$N_s \int d\boldsymbol{\xi}_s F_s A_h.$$

Another point of interest is to observe that the correction deduced from the result (5–99) permits us to replace the first approximation for H_s^t given in (5–97) by the better one

$$(H_s^t)_{II} = -\frac{r_D^2 F_s}{2\pi^2(1 + \nu^2 r_D^2)\kappa T}. \tag{5–105}$$

Evidently this still is a "static" approximation, not containing any reference to the relative velocity vector $\mathbf{w} = \boldsymbol{\xi}_s = \mathbf{u}_t$. Nevertheless, by applying the inverse Fourier transform, this expression can be used to obtain an interesting approximation for F'_{st} and for F_{st}. If the result is divided by F_s, the quantity $F_{st}/F_s = F_t + F'_{st}/F_s$ gives the probability that a particle t with arbitrary velocity components ξ_{th} will be found at a location $\mathbf{x}_t$, when it is known that a particle s is present at $\mathbf{x}_s$. We can use this result to calculate the electrostatic potential at an arbitrary point $\mathbf{x}$ in the neighborhood of $\mathbf{x}_s$ due to all particles t in the neighborhood of the particle s. The outcome of the calculation appears to be

$$-e_s\left\{\frac{1}{r} - \frac{\exp(-r/r_D)}{r}\right\}, \tag{5–106}$$

where r represents the absolute value of the distance of the point $\mathbf{x}$ from the point $\mathbf{x}_s$. The result means that around the particle s we have obtained the Debye potential field.

In principle, it should be possible to use a more sophisticated expression for the function H_s^t, in order to arrive at a potential field which brings into evidence the effect of the relative speed of the particle s. There must be a distortion of the Debye cloud connected with this speed and, as Gross remarked, at very high relative speeds the particle s appears to "shake off" the cloud.

It must be observed that in the deduction of Eq. (5–106) it has been assumed that the mean charge density over the volume of the Debye sphere is zero; if this had not been done, it would have been found that there remains a term connected with the mean space charge. Such a space charge, however, requires other methods for its proper handling.

In the discussion, Gross asked where the irreversibility has come into the treatment, since the equation which lies at the basis of Eqs. (5–83) and (5–87) is a Liouville equation which in itself describes reversible phenomena. In reply Tchen pointed out that in working with these equations we do not start from precisely defined initial conditions, but assume initial conditions described by random data, as, for instance, $F'_{st}(-\infty) = 0$. This introduces irreversibility, since we cannot make the same assumption for $t = +\infty$.

5–23 Plasma oscillations. Collective coordinates. Next, we turn our attention to the problem of oscillations occurring within a plasma. (See Sections 1–5, 2–3, 3–4, 4–8, 4–9, and 4–10.) From papers which have appeared in recent literature [52] it will be seen that various aspects of this subject have been investigated, often to great depth, but there is not to be found so far a comprehensive treatment which brings all features into proper perspective. Several of these features need the application of the Boltzmann equation for their treatment; they cannot be treated satisfactorily from the continuum equations alone. It is in particular the anisotropic character of the pressure tensor which is involved. Another point is the trapping of electrons by waves.

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52. From the many papers referring to plasma oscillations the following may be mentioned:

D. Bohm and E. Gross, papers mentioned in Ref. 13.

D. Bohm and D. Pines, A collective description of electron interactions I. Magnetic interactions, *Phys. Rev.* **82,** 625–634 (1951); II. Collective *vs.* individual particle aspect of the interactions, *Phys. Rev.* **85,** 338–353 (1952).

N. G. van Kampen, On the theory of stationary waves in plasmas, *Physica* **21,** 949–963 (1955).

I. B. Bernstein, J. M. Greene and M. D. Kruskal, paper mentioned in Ref. 16.

I. B. Bernstein, Waves in a plasma in a magnetic field, *Phys. Rev.* **109,** 10–21 (1958).

Franz Kahn mentioned the possibility of describing plasma behavior in terms of Fourier components of the charge density. Consider a single charge e_1 at a point $\mathbf{a}_1$ within a rectangular volume V, with sides X_0, Y_0, Z_0. It contributes a charge density

$$\rho_1(\mathbf{r}) = e_1\,\delta(\mathbf{r} - \mathbf{a}_1) = \frac{e_1}{V} + \frac{2e_1}{V}\sum_s \cos \mathbf{k}_s \cdot (\mathbf{r} - \mathbf{a}_1),$$

the summation being made over all lattice points in half of the wave-number space. A typical lattice point is

$$\mathbf{k}_s = (2\pi s_1/X_0, 2\pi s_2/Y_0, 2\pi s_3/Z_0),$$

where s_1, s_2, s_3 are integers. With N charges $e_1, \ldots, e_N$ in the volume, the charge density becomes

$$\rho(\mathbf{r}) = \sum_n e_n\,\delta(\mathbf{r} - \mathbf{a}_n) = \frac{Ne}{V} + \frac{2e}{V}\sum_s \mu_s \cos(\mathbf{k}_s \cdot \mathbf{r} - \alpha_s),$$

where

$$\mu_s \cos \alpha_s = \sum_n e_n \cos(\mathbf{k}_s \cdot \mathbf{a}_n),$$

$$\mu_s \sin \alpha_s = \sum_n e_n \sin(\mathbf{k}_s \cdot \mathbf{a}_n).$$

If the charges were randomly distributed, the expectation value of each μ_s^2 would be $\frac{1}{2}\sum e_n^2$. But in thermodynamic equilibrium the electrostatic interactions between particles do not allow the charge distribution to be entirely random. We find that $\langle \mu_s^2 \rangle$ is less than $\frac{1}{2}\sum e_n^2$ for wave numbers smaller than about $2\pi/\lambda_D$, where λ_D is the Debye wavelength.

Knowing the different μ's, one can work out correlation functions for the positions of the charges in the gas. The effect described here leads to a decrease in the scattering cross section of the electrons for electromagnetic radiation. This result has an application in astrophysics: the reduction in the cross section entails a decrease in the opacity of the material near the centers of the hotter main sequence stars [53].

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53. A. I. AKHIEZER, I. G. PROKHODA, and A. G. SITENKO, Scattering of electromagnetic waves in a plasma (Russian text), *J. Zhur. Eksptl. i Teort. Fiz.*; Am. Inst. Phys. translation in *Soviet Physics JETP* **33**: 6, 576–582 (1957); F. D. KAHN, Long range interactions in ionized gases in thermal equilibrium, *Astrophys. J.* **129**, No. 1, 205–216 (Jan. 1959).

In connection with these remarks, Thompson referred to some problems treated by Edwards [54] concerning the evaluation of the equilibrium properties of a plasma. In this work it was necessary to calculate the partition function

$$Z = \int \cdots \int dx_1^3\, dv_1^3 \cdots dx_N^3\, dv_N^3 \exp\left(\frac{-H}{\kappa T}\right),$$

where

$$H = \sum_i \tfrac{1}{2}mv_i^2 + \sum_{i \neq j} \frac{e^2}{r_{ij}}.$$

The integration with respect to the velocities is trivial and we are left with integrals over the configuration space. If the potentials were short range, this could be treated by a cluster expansion, as is used by Mayer and others. Edwards' technique is to split the potential effectively as follows:

$$\phi = \frac{1}{r} = \frac{\exp(-r/h)}{r} + \left\{\frac{1}{r} - \frac{\exp(-r/h)}{r}\right\} = \phi_s + \phi_l,$$

and to use the Mayer expansion for ϕ_s. For ϕ_l he uses the collective coordinates of Bohm and Pines, that is, the quantities

$$\rho_k = \sum_j \exp(i\mathbf{k} \cdot \mathbf{x}_j).$$

This requires the introduction of the Jacobian $J(\rho_k, \mathbf{x}_j)$, which is found to be proportional to

$$\exp(-\rho\rho^*/2N),$$

after which the integrals can be obtained. There remains the problem of determining the correct value of h, which is so far an arbitrary parameter. This is done by using a minimum principle. The free energy is calculated, and Edwards insists that one must have $\partial F/\partial h = 0$. This leads to an integral equation for h, which gives the result that $h \sim \lambda_D$, the Debye length.

Then Gross took up the subject, referring to work he did with Bohm, described briefly in reference [6]. Here a completely collective approach was used. A few of the equations are given below.

The Hamiltonian for the motion of the electrons is taken in the form

$$\mathcal{H} = \sum_i (p_i^2)/2m + \tfrac{1}{2}\sum_{ij} V(\mathbf{x}_i - \mathbf{x}_j) \qquad (5\text{–}107)$$

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54. S. F. EDWARDS, A variational calculation of the equilibrium properties of a classical plasma, *Phil. Mag.* (8th ser.) **3**, 119–124 (1958).

where V is the potential energy of two particles at the positions $\mathbf{x}_i$, $\mathbf{x}_j$. The summation is over all i, j, from 1 through N, with $i \neq j$. A contact transformation [55] is introduced as described by the equations

$$\rho_k = \frac{\partial G}{\partial \pi_k}, \qquad \mathbf{p}_i = \frac{\partial G}{\partial \mathbf{x}_i}, \tag{5–108a}$$

with

$$G = \sum_{i,k} \pi_k \exp(-i\mathbf{k}\cdot\mathbf{x}_i)\cdot \tag{5–108b}$$

Here $\mathbf{k}$ (and similarly $\mathbf{l}$ in later equations) is summed over all integral points $\pm k_1$, $\pm k_2$, $\pm k_3$, with the exception of $k = 0$ (which is necessary in connection with the compensating background charge of opposite sign). From Eqs. (5–108a) we obtain

$$\rho_k = \sum_i \exp(-i\mathbf{k}\cdot\mathbf{x}_i), \qquad \mathbf{p}_i = -i\sum_k \mathbf{k}\pi_k \exp(-i\mathbf{k}\cdot\mathbf{x}_i). \tag{5–108c}$$

The transformation has the property that the Hamiltonian form of the equations of motion is retained. When expressed in the new variables ρ_k, π_k, the Hamiltonian becomes

$$\mathcal{H} = -\frac{1}{2m}\sum_{k,l}(\mathbf{k}\cdot\mathbf{l})\pi_k\pi_l\rho_{k+l} + \sum_k \frac{2\pi e^2}{k^2}\rho_k\rho_{-k}. \tag{5–109}$$

By making the transformation (5–108a), the difficulties of the N-body problem are split in two parts. One is that of solving the equations of motion for the new variables, which have the form

$$\begin{aligned}\dot{\rho}_k &= -\frac{1}{m}\sum_h (\mathbf{k}\cdot\mathbf{h})\pi_h\rho_{k+h},\\ \dot{\pi}_k &= \frac{1}{2m}\sum_h \{(\mathbf{k}-\mathbf{h})\cdot\mathbf{h}\}\pi_{h-k}\pi_h - \frac{4\pi e^2}{k^2}\rho_{-k}.\end{aligned} \tag{5–110}$$

The second part is to solve the implicit equations (5–108c) to get back information about the particles in terms of the $\mathbf{x}_i$ and the $\mathbf{p}_i$. This division is similar to that of the general transformation theory of dynamics. There, by making the transition to angle and action variables, the first part becomes trivial and the second part hard. But many properties of systems can be ascertained without analysis of the second part. For the plasma

55. See, e.g., E. T. WHITTAKER, *Analytical Dynamics*, Cambridge Univ. Press, England, 1917, pp. 292–296.

the first part, i.e., solving the equation for ρ_k and π_k, is not easy. But again many questions do not require inverting Eqs. (5–108c). The collective response of the plasma can be discussed entirely in terms of the quantities built up from the ρ_k and π_k. One then deals with variables particularly appropriate for correlation and turbulence theory.

It may be interesting to add an approximate treatment of Eqs. (5–110), following Bohm and Pines [56], although here a mixed representation is used and not the approach emphasized by Gross. From the two equations (5–110) we obtain

$$\begin{aligned}\ddot{\rho}_k = \frac{1}{m^2}\sum_{l,h}(\mathbf{k}\cdot\mathbf{l})[(\mathbf{k}+\mathbf{l})\cdot\mathbf{h}]\pi_l\pi_h\rho_{k+l+h} &- \\ -\frac{1}{2m^2}\sum_{l,h}(\mathbf{k}\cdot\mathbf{l})[(\mathbf{l}-\mathbf{h})\cdot\mathbf{h}]\pi_{l-h}\pi_h\rho_{k+l} &+ \\ +\frac{4\pi e^2}{m}\sum_l \frac{(\mathbf{k}\cdot\mathbf{l})}{l^2}\rho_{k+l}\rho_{-l}.&\end{aligned} \tag{5–111a}$$

The single sum occurring in the last line of this formula can be written

$$\frac{4\pi e^2}{m}\sum_{i,l}\frac{(\mathbf{k}\cdot\mathbf{l})}{l^2}\exp[-i(\mathbf{k}+\mathbf{l})\mathbf{x}_i]\rho_{-l}.$$

Bohm and Pines, in treating a problem of this type, assume that the only contributions of importance in this sum are those with $\mathbf{l} = -\mathbf{k}$. The sum then reduces to

$$-\frac{4\pi N e^2}{m}\rho_k.$$

The double sums can be taken together and reduced to

$$\sum_{l,h}(\mathbf{k}\cdot\mathbf{l})(\mathbf{k}\cdot\mathbf{h})\pi_l\pi_h\rho_{k+l+h},$$

which can be transformed into

$$-\sum_i (\mathbf{k}\cdot\mathbf{p}_i)^2\exp(-i\mathbf{k}\cdot\mathbf{x}_i)\cdot$$

Here Bohm and Pines assume that we may replace $(\mathbf{k}\cdot\mathbf{p}_i)^2$ by a mean value. If we take this to be $(k^2/3)(p_i^2)_{\text{mean}} = k^2\kappa T m$, we arrive at the following equation:

$$\ddot{\rho}_k = -\left\{\frac{4\pi N e^2}{m} + k^2\frac{\kappa T}{m}\right\}\rho_k. \tag{5–111b}$$

56. See BOHM and PINES, paper II mentioned in Ref. 52, pp. 340–341.

This leads to oscillations with a frequency

$$\omega = \left(\frac{4\pi Ne^2}{m} + k^2 \frac{\kappa T}{m}\right)^{1/2}. \tag{5–112}$$

As is well known, this is an approximation for space-charge waves (longitudinal plasma oscillations) in the elementary theory [57].

57. An accurate treatment leads to a factor 3 before $k^2(\kappa T/m)$. See BOHM and PINES, paper mentioned in Ref. 13, p. 1867; P. L. BHATNAGAR, M. KROOK, and D. H. MENZEL, Preliminary report of the committee on dynamics of ionized media, Harvard College Observatory (1952), p. 25; J. M. BURGERS, paper mentioned in Ref. 50, p. 32.

CHAPTER 6

CONTINUUM PLASMA DYNAMICS

HANS W. LIEPMANN and JULIAN D. COLE, *Editors*

6–1 Introduction (based on remarks by George Batchelor). The people who are interested in the physics of gases and plasmas can be divided, broadly speaking, into two groups. The first group constitutes the continuum or field specialists who like to think of gases, ions, and electrons as smeared over the whole of space. They are in their element when the mean free path is small compared with all other lengths occurring in the plasma. The other group comprises those who are equally in their element when the opposite situation holds. These people come with the point of view that space is mostly vacant and the occurrences of particles are such special events that they must be treated with the individual respect that is their due. In the present chapter, we shall be concerned with the accomplishments and aspirations of the first group.

As a matter of fact, the distinction between the two points of view is not so clear as was once supposed. Aerodynamicists have become quite familiar with phenomena like shock waves, where large changes occur in one or two mean free paths. They have also learned to handle the problems of flight at low densities, where the mean free path is comparable to the thickness of the boundary layer. The word "continuum" is now being used in an extended sense to describe not only the physical model that is being used, but also the approach or the attitude toward the problem. If the problem is formulated in terms of a relatively small number of dependent variables, then the continuum approach is being used. If the number of dependent variables is large, the particle approach is being used. Since only a reasonable amount of information can be digested from a given problem, the dependent variables of the particle approach must finally be condensed by some averaging process. Thus the distinction ultimately resides in whether the average occurs before or after the analysis, i.e., whether statistics precede or follow mechanics.

Obviously there exist regimes for which one of the approaches becomes particularly simpler than the other. For example, one would hardly think of dealing with water waves from the particle point of view, or with continuum mechanics in molecular beam experiments. In the dynamics of conducting gases there does exist, however, a broad realm of conditions where both approaches can be applied and where the choice is often a matter of taste or of previous experience and skill. The continuum equa-