"Plasma Physics", IAEA (1965)

DERIVATION OF MACROSCOPIC EQUATIONS

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Since we shall be concerned with collections of large numbers of particles interacting with their self-created and/or externally imposed electromagnetic fields and since it is in general a problem of prohibitive difficulty to follow the detailed motion of all the particles we must rest content to describe the behaviour of the plasma system in some average or statistical sense.

It is often desirable to be content to describe the plasma in terms of the crudest of statistical theories, essentially a hydrodynamic description in terms of mean number density, mean velocity, pressure, etc., modified to include electromagnetic effects.

The value of working at this low level of description is that one can quickly get an insight into the behaviour of the plasma and obtain a large body of qualitative results which are, in general, only somewhat modified by more complete (and thus complex) statistical theories.

We shall now suppose the totally ionized plasma to consist of electrons and positive ions (of one species), masses m^- and m^+, and charges $e^-(=-e)$ and e^+, respectively. The generalization to more complex systems with ions in various stages of ionization is straightforward as long as the internal dynamics of an ion is negligible.

For each particle of our system we have an equation of motion

$$m_n \dot{\vec{v}}_n = e_n \left[\vec{E}(\vec{r}_n, t) + \frac{\vec{v}_n}{c} \times \vec{B}(\vec{r}_n, t) \right], \tag{1a}$$

$$\dot{\vec{r}}_n = \vec{v}_n . \tag{1b}$$

(We have written down the non-relativistic equation of motion and indeed for most cases of interest this description is adequate. In some problems, however, e.g. synchrotron radiation, the relativistic description must be invoked and we shall write down for completeness the equation of motion of a single particle in relativistic form:

$$\frac{d}{dt}(\gamma m \vec{v}) = e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right), \tag{2}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ and m is the rest mass, but we shall not pursue this equation further at this time.)

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The equations of motion for the electromagnetic field, Maxwell's equations, are

$$\frac{\partial \vec{B}}{\partial t}(\vec{r},t) = -c\vec{\nabla}\times\vec{E}(\vec{r},t), \tag{3}$$

$$\frac{\partial \vec{E}}{\partial t} = c\vec{\nabla}\times\vec{B} - 4\pi\vec{J}(\vec{r},t)\,, \tag{4}$$

$$\vec{\nabla}\cdot\vec{B} = 0, \tag{5}$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\sigma(\vec{r},t). \tag{6}$$

These equations relate the development of the electromagnetic field intensities $\vec{E}$, $\vec{B}$ in terms of the sources, charge density σ and current density $\vec{J}$. Notice that we have made no distinction between $\vec{E}$ and $\vec{D}$, $\vec{B}$ and $\vec{H}$ since we shall keep full track of our internal sources. It is only when one abandons some of the sources appearing in Eqs. (4) and (6) that the distinction must arise.

We now must relate the sources σ, $\vec{J}$ to the dynamical motion of the plasma. Now instead of relating the electromagnetic fields to the exact dynamical motion of the particles,

$$\sigma(\vec{r},t) = \sum_n e_n\delta\,[\vec{r}-\vec{r}_n(t)]\,, \tag{7}$$

$$\vec{J}(\vec{r},t) = \sum_n e_n\vec{v}_n\delta[\vec{r}-\vec{r}_n(t)] \tag{8}$$

we shall embark on the statistical description.

We presuppose the existence of a distribution function for each species $f^+(\vec{r},\vec{v},t)$ and $f^-(\vec{r},\vec{v},t)$. The distribution function has the meaning that $f(\vec{r},\vec{v},t)d^3r\,d^3v$ ($d^3r \equiv dx\,dy\,dz$ and $d^3v = dv_x dv_y dv_z$) represents the probable number of particles of each type in the volume element $d^3r\,d^3v$ at the point $(\vec{r},\vec{v})$ in the six-dimensional phase space (μ-space). The particle mean number density and mean velocity of each type are defined by the zero'th and first moment of these distribution functions with respect to velocity:

$$n^\pm(\vec{r},t) = \int f^\pm(\vec{r},\vec{v},t)d^3v, \tag{9}$$

$$\vec{u}^\pm(\vec{r},t) = \frac{1}{n^\pm(\vec{r},t)}\int \vec{v}f^\pm\,d^3v. \tag{10}$$

We now take the sources σ, $\vec{J}$ to be given by

$$\sigma(\vec{r},t) = \sum_{+,-} en, \tag{11}$$

$$\vec{J} = \sum_{+,-} en\vec{u}\,. \tag{12}$$

The fields $\vec{E}$ and $\vec{B}$ computed from these averages σ, $\vec{J}$ are called the self-consistent, or average internal fields. Any sample particle is subject not only to the self-consistent and/or external fields but also to rapidly fluctuating micro-fields, i.e. forces due to encounters with neighbouring particles in the configuration space. Let us for the time being ignore these micro-fields and determine the laws of motion for $f^\pm(\vec{r},\vec{v},t)$.

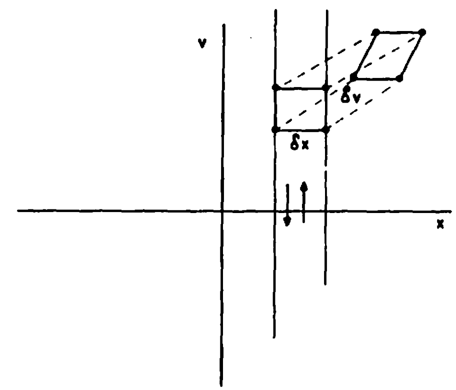

Fig. 1

Motion of volume element in phase space

Consider a small element of volume $\delta\Omega = d^3r\,d^3v$ in the phase space (Fig. 1). The number of phase points associated with particles of either species in this element is $f^\pm(\vec{r},\vec{v},t)\delta\Omega$. During the course of time the particles corresponding to these phase points move over the phase space but at all times keeping the same number of particles in $\delta\Omega$, in virtue of the fact that neighbouring particles execute neighbouring motions. (The external and/or self-consistent fields are "smooth" over $\delta\Omega$.) Thus,

$$\frac{d}{dt}\,[f^\pm(\vec{r},\vec{v},t)\delta\Omega] = 0, \tag{13}$$

or

$$\frac{df}{dt}\delta\Omega + f\frac{d(\delta\Omega)}{dt} = 0. \tag{13a}$$

We shall now show $d/dt(\delta\Omega) = 0$.

We shall give this derivation of the law of conservation of extension in phase in more general terms, applicable also to the Liouville equation.

Consider the laws of motion for a system of n degrees of freedom in their first order form

$$\dot{z}_i = g_i(z_1, z_2, \ldots, z_n),$$

where z_i may be a co-ordinate or a velocity. Consider the extension in phase

$$\delta\Omega = \prod_i \delta z_i .$$

Then

$$
\begin{aligned}
(\delta\Omega)^{\cdot}/\delta\Omega &= \sum_i \frac{\delta \dot{z}_i}{\delta z_i} \\
&= \sum_i [g_i(z_1, \ldots, z_i + \delta z_i, \ldots, z_n) - g_i(\ldots, z_i, \ldots)]/\delta z_i \\
&= \sum_i \frac{\partial g_i}{\partial z_i} + \text{higher order terms} \\
&\equiv \vec{\nabla}_n \cdot \vec{g} + \text{higher order terms.}
\end{aligned}
$$

For Hamiltonian systems $\vec{\nabla}_n \cdot \vec{g} = 0$.

In our present case, since $\dot{\vec{r}} = \vec{v}$, $\dot{\vec{v}} = \frac{e}{m}\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$,

$$\frac{\partial}{\partial \vec{r}} \cdot \vec{v} = 0; \quad \frac{\partial}{\partial \vec{v}} \cdot \left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right] = \vec{\nabla}_v \cdot \vec{E} + \frac{1}{c}\vec{B} \cdot (\vec{\nabla}_v \times \vec{v}) - \frac{1}{c}\vec{v} \cdot \vec{\nabla}_v \times \vec{B} = 0,$$

where the three terms of the last expression are equal to zero. We thus have

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\vec{r}} \cdot \frac{\partial f}{\partial \vec{r}} + \dot{\vec{v}} \cdot \frac{\partial f}{\partial \vec{v}} = 0, \tag{14}$$

or

$$\frac{\partial f^{\pm}}{\partial t} + \vec{v} \cdot \frac{\partial f^{\pm}}{\partial \vec{r}} + \frac{e^{\pm}}{m^{\pm}}\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) \cdot \frac{\partial f^{\pm}}{\partial \vec{v}} = 0. \tag{15}$$

If we now reconsider the fluctuating micro-fields, i.e. collisions between particles, then particles are continually transferred from one element $\delta x \, \delta v$ to another element in the same strip δx, so that a term must be added

to the right-hand side of Eq. (15) to record the balance between particles entering and leaving a given volume element of the phase space because of collisions. We then write our kinetic equation for either species as

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \frac{e}{m}\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) \cdot \frac{\partial f}{\partial \vec{v}} = \left(\frac{\delta f}{\delta t}\right)_c . \tag{16}$$

The detailed structure and properties of the collision term is discussed in other lectures and we shall merely posit at this time any properties of this term that we shall need.

Let us now take moments of this equation with respect to velocity, i.e. multiply by $1, \vec{v}, \vec{v}\vec{v}, \vec{v}\vec{v}\vec{v}$, etc., and integrate over velocity space. For the zero'th moment

$$\int d^3v \frac{\partial f}{\partial t}(\vec{x}, \vec{v}, t) = \frac{\partial}{\partial t}\int d^3v f = \frac{\partial n}{\partial t}(\vec{r}, t), \tag{17}$$

$$\int d^3v \, \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} = \frac{\partial}{\partial \vec{r}} \cdot \int d^3v \, \vec{v} f = \vec{\nabla} \cdot (n\vec{u}), \tag{18}$$

$$\int d^3v \frac{e}{m}\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) \cdot \frac{\partial f}{\partial \vec{v}} = \frac{e}{m}\int d^3v \vec{\nabla}_{\vec{v}} \cdot \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) = 0, \tag{19}$$

where we have assumed f vanishes sufficiently strongly for large velocities so that all surface integrals in velocity space vanish. Since individual collisions conserve number, the collision term has no zero'th moment.

We thus arrive at the equation of number continuity for each species

$$\frac{\partial n^{\pm}}{\partial t} + \vec{\nabla} \cdot (n^{\pm}\vec{u}^{\pm}) = 0 . \tag{20}$$

The equation of charge continuity,

$$\frac{\partial \sigma}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \tag{21}$$

and mass continuity,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{U}_0) = 0 \tag{22}$$

follow from Eq. (20) by multiplying by the charge of each species and summing over species, and by multiplying by the mass of each species and summing over species, respectively. Here

$$\rho = \sum_{+,-} mn \tag{23}$$

$$\vec{U}_0 = \frac{1}{\rho}\sum_{+,-} m n \vec{u} \tag{24}$$

are the mass density and velocity of centre of mass, respectively.

To arrive at the macroscopic equations of motion we take the first moment, i.e. multiply by $m\vec{v}$ and integrate over velocities:

$$m\int d^3v\, \vec{v}\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}(mn\vec{u}), \tag{25}$$

$$m\int d^3v\, \vec{v}\vec{v}\cdot\frac{\partial f}{\partial \vec{r}} = \vec{\nabla}\cdot\left(m\int d^3v\, \vec{v}\vec{v} f\right), \tag{26}$$

$$m\int d^3v\, \vec{v}\left[\frac{e}{m}\left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}\right)\cdot\frac{\partial f}{\partial \vec{v}}\right] = -e\int d^3v \left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}\right) f$$

$$= -en\left(\vec{E}+\frac{\vec{u}}{c}\times\vec{B}\right) \tag{27}$$

$m\int d^3v\, \vec{v}\left(\frac{\delta f}{\delta t}\right)_c = p_{ss'}$, momentum transferred per unit time by collisions with opposite species. (28)

(Like-like collisions produce no net momentum change in virtue of Newton's third law.)

If we now define the stress dyadic (or tensor) for each species as

$$\vec{P}^{\pm} = m^{\pm}\int d^3v(\vec{v}-\vec{U}_0)(\vec{v}-\vec{U}_0)f^{\pm}$$

$$= m^{\pm}\int d^3v\, \vec{v}\vec{v} f^{\pm} - m^{\pm}n^{\pm}\vec{u}^{\pm}\vec{U}_0 - m^{\pm}n^{\pm}\vec{U}_0\vec{u}^{\pm} + m^{\pm}n^{\pm}\vec{U}_0\vec{U}_0 \tag{29}$$

we then have

$$\frac{\partial}{\partial t}(mn\vec{u}) + \vec{\nabla}\cdot[m^{\pm}n^{\pm}(\vec{u}^{\pm}\vec{U}_0+\vec{U}_0\vec{u}^{\pm}-\vec{U}_0\vec{U}_0)] + \vec{\nabla}\cdot\vec{P}^{\pm} = e^{\pm}n^{\pm}\left(\vec{E}+\frac{\vec{u}^{\pm}}{c}\times\vec{B}\right). \tag{30}$$

If we sum over both species we have, using Eq. (24), and with $\vec{P}\equiv\vec{P}^{+}+\vec{P}^{-}$,

$$\frac{\partial}{\partial t}(\rho\vec{U}_0) + \vec{\nabla}\cdot\vec{P} + \vec{\nabla}\cdot(\rho\vec{U}_0\vec{U}_0) = \sigma\vec{E}+\frac{\vec{J}}{c}\times\vec{B}, \tag{31}$$

where again we have used the third law, $p_{ss'} = -p_{s's}$. If we note

$$\vec{\nabla}\cdot(\rho\vec{U}_0\vec{U}_0) = \vec{U}_0\vec{\nabla}\cdot(\rho\vec{U}_0) + \rho\vec{U}_0\cdot\vec{\nabla}\vec{U}_0 \tag{32}$$

and employ the equation of mass continuity (22), we have

$$\rho\frac{d\vec{U}_0}{dt} = -\vec{\nabla}\cdot\vec{P} + \sigma\vec{E} + \frac{\vec{J}}{c}\times\vec{B}, \tag{33}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{U}_0\cdot\vec{\nabla}. \tag{34}$$

If there are external body forces, or other average internal forces (e.g. gravity), they appear naturally on the right-hand side of Eq. (33).

(The definition (29) of the stress tensor is not universal, but is appropriate when both species are very closely Maxwellian relative to the centre of mass velocity and at the same temperature.) Often in plasmas macroscopic phenomena take place on a time scale fast compared to that on which any significant alteration of the distribution function due to collisions can take place and another definition of pressure is often appropriate.

$$\vec{P}^{\pm} = m^{\pm}\int d^3v(\vec{v}-\vec{u})(\vec{v}-\vec{u})f. \tag{35}$$

The analogue of Eq. (30) with this definition of the stress tensor is

$$m^{\pm}n^{\pm}\left(\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\vec{\nabla}\vec{u}\right) = e^{\pm}n^{\pm}\left(\vec{E}+\frac{\vec{u}}{c}\times\vec{B}\right) - \vec{\nabla}\cdot\vec{P}^{\pm} + p_{ss'}. \tag{36}$$

We now notice that

$$\tfrac{1}{2}\vec{P}^{\pm}:\vec{I} = \tfrac{1}{2}\mathrm{Trace}\,\vec{P}^{\pm} = \tfrac{1}{2}m\int d^3v(\vec{v}-\vec{U}_0)(\vec{v}-\vec{U}_0)f^{\pm} \tag{37}$$

represents the mean kinetic energy of the system relative to the centre of mass and may be referred to as the internal energy of the system or the thermal energy of the gas. We may define a generalized temperature through

$$\tfrac{1}{2}\mathrm{Trace}\,\vec{P}^{\pm} = \tfrac{3}{2}N^{\pm}\Theta, \tag{38}$$

where $\Theta = kT$ (k is here Boltzmann's constant). Again, it is possible and sometimes desirable to use two separate temperatures for the separate species.

There are two important points to notice at this stage. Equations (22) and (33) together with Maxwell's Eqs. (3)-(6), although in some sense exact, do not form a closed set of macroscopic equations. The Maxwell equations governing the time evolution of the electromagnetic fields involve the charge and current densities. To find the time evolution of these quantities we could use Eqs. (20) and (30) for each species but Eq. (30) involves the knowledge of $p_{ss'}$, i.e. properties of the collision term. Equations (21), (22), and (33) are not enough since we are still one vector equation short. We could write an equation for $\partial\vec{J}/\partial t$ (the so-called generalized Ohm's law) but this involves the $p_{ss'}$ again.

The second point is that we still have no equations for $\vec{P}$. If we compute the equation of motion for $\vec{P}$ by ascending the moment ladder further we find for each species

$$\frac{\partial \vec{P}}{\partial t} + \vec{\nabla}\cdot(\vec{Q} + \vec{U}_0\vec{P}) + \vec{P}\cdot\vec{\nabla}\vec{U}_0 + (\vec{P}\cdot\vec{\nabla}\vec{U}_0)^T + mn\frac{d\vec{U}_0}{dt}\vec{U} + mn\vec{U}\frac{d\vec{U}_0}{dt}$$

$$+\frac{e}{c}(\vec{B}\times\vec{P} - \vec{P}\times\vec{B}) - en\vec{U}\left(\vec{E} + \frac{\vec{U}_0}{c}\times\vec{B}\right) - e\left(\vec{E} + \frac{\vec{U}_0}{c}\times\vec{B}\right)n\vec{U} = \left(\frac{\delta\vec{P}}{\delta t}\right)_c, \quad (39)$$

where

$$\vec{Q} = m\int d^3v(\vec{v} - \vec{U}_0)(\vec{v} - \vec{U}_0)(\vec{v} - \vec{U}_0)_f \quad (40)$$

is the heat flow triadic, and

$$\vec{U} = \frac{1}{n}\int d^3v(\vec{v} - \vec{U}_0)f \quad (41)$$

is the mean velocity of each species relative to the centre of mass, and

$$(\vec{P}\cdot\vec{\nabla}\vec{U}_0)^T$$

is the transpose of the dyadic $(\vec{P}\cdot\vec{\nabla}\vec{U}_0)$. In this last equation we have omitted the species label + or − . If we now sum over species we have

$$\frac{\partial \vec{P}}{\partial t} + \vec{\nabla}\cdot(\vec{Q} + \vec{U}_0\vec{P}) + \vec{P}\cdot\vec{\nabla}\vec{U}_0 + (\vec{P}\cdot\vec{\nabla}\vec{U}_0)^T$$

$$+\sum_{+,-}\frac{e}{mc}(\vec{B}\times\vec{P} - \vec{P}\times\vec{B}) + (\vec{J} - \sigma\vec{U}_0)\left(\vec{E} + \frac{\vec{U}_0}{c}\times\vec{B}\right), \quad (42)$$

where we have made use of the fact that

$$\sum mn\vec{U} = 0 \quad (43)$$

and

$$\sum en\vec{U} = \vec{J} - \sigma\vec{U}_0. \quad (43a)$$

By taking one-half the trace of Eq. (42) we obtain the equation of energy balance

$$\frac{\partial}{\partial t}\left(\frac{3}{2}n\Theta\right) + \vec{\nabla}\cdot\left(\vec{q} + \frac{3}{2}n\Theta\vec{U}_0\right) + \vec{P}:\vec{\nabla}\vec{U}_0 - (\vec{J} - \sigma\vec{U}_0)\left(\vec{E} + \frac{\vec{U}_0}{c}\times\vec{B}\right) = 0. \quad (44)$$

Here

$$q = \frac{1}{2}\sum_{+,-} m\int d^3v(\vec{v} - \vec{U}_0)^2(\vec{v} - \vec{U}_0)f \quad (45)$$

is the heat flow vector and represents the flow of internal energy relative to the centre of mass. The term involving $(\vec{B}\times\vec{P} - \vec{P}\times\vec{B})$ has zero trace. Here n is the sum of the number densities. The collision term gives no contribution because collisions preserve energy.

We see from Eq. (42) that the equation governing the time rate of change involves the heat flow tensor. There is no rigorous way to close the moment hierarchy.

There are two important limiting situations where the moment scheme may be closed:

(a) The first is where collisions dominate. This situation is most often realized in weakly ionized gases, where collisions with neutrals dominate, but sometimes even in totally ionized gas this situation obtains. Here one can make a development along the lines of the Chapman-Enskog theory. One arrives at a set of transport coefficients, resistivity coefficients, coefficient of thermal conductivity, coefficient of viscosity, etc., which relate the fluxes such as current field, thermal gradients and mass velocity gradients. This will be discussed in other papers.

(b) In these situations where to lowest approximation collisions are negligible (see chart), and if the characteristic frequencies are high and/or wave numbers are small such that, crudely, $L/T \approx \omega/k \gg v_{th}$ (the so-called "Low Temperature Approximation"), then to lowest approximation the pressure may be dropped from the equation of motion. The next approximation taking into account thermal corrections consists in dropping the heat flow term from the equation for the pressure development (Eq. (42)) which leads in interesting situations to certain adiabatic laws for the pressure development.

There is one additional comment preparatory to the mutilation of these equations as we begin our study of plasma properties, that is their non-linearity. A few idealized situations have been studied which capture this feature. The usual procedure is to examine small departures from some known equilibrium or steady flow. Unfortunately, far too often these situations prove to be unstable, and to examine their fate (turbulence of "hash") the non-linearity must be invoked. Only very recently have we come to get even the slightest grip on these problems, and finding suitable techniques for handling them is one of the outstanding current problems in plasma physics and will be discussed later.

We have just outlined the two essential difficulties in closing the macroscopic moment equations:

(a) If we think in terms of the two macroscopic velocities $\vec{u}^+$, $\vec{u}^-$ or equivalently in terms of $\vec{J}$, $\vec{U}_0$, there remains something to do with the term $p_{ss'}$, the momentum transferred per unit time by collisions with opposite species.

(b) How shall the pressure be determined, when, perhaps, neither the "Low-Temperature" nor collision-dominated situation prevails?

Let us return to Eq. (36) for each species, divide by m^s, multiply by e^s, and sum over species. With $e^+ = Ze$, $e^- = -e$, we then have, utilizing Eqs. (11), (12), (23), and (24),

$$\frac{\partial J'}{\partial t} + \sum_s [\vec{u}_s \vec{\nabla}\cdot(n_s e_s \vec{u}_s) + e_s n_s \vec{u}_s \cdot \vec{\nabla}\vec{u}_s] = e^2\left(\frac{Z^2 n^+}{m^+} + \frac{n^-}{m^-}\right)\vec{E}$$

$$+ \frac{e^2}{c(m^+ + Zm^-)}\rho\left(\frac{Z^2}{m^+} + \frac{Z}{m^-}\right)\vec{U}_0\times\vec{B} + \frac{e}{c(m^+ + Zm^-)}\left(Z^2\frac{m^-}{m^+} - \frac{m^+}{m^-}\right)\vec{J}\times\vec{B}$$

$$- e\left(\frac{Z}{m^+}\vec{\nabla}\cdot\vec{P}'^+ - \frac{1}{m^-}\vec{\nabla}\cdot\vec{P}'^-\right) + e\left(\frac{Z}{m^+} + \frac{1}{m^-}\right)p_{+-}. \qquad (46)$$

Now, under the assumptions (1) $m^- \ll m^+$, (2) all terms quadratic in the $\vec{u}_s$ and their derivatives may be neglected (generally valid if $|\vec{u}_s| \ll (p/\rho)^{1/2}$, $(B^2/4\pi\rho)^{1/2}$, i.e. if macroscopic velocities are $\ll$ sound speed or hydromagnetic speed), (3) $n^+ \approx n^-/Z$, and (4) $P'^- \sim P'^+$, then this equation reduces to

$$\frac{m^-}{e^2 n^-}\frac{\partial \vec{J}}{\partial t} = \vec{E} + \frac{\vec{U}_0}{c}\times\vec{B} - \frac{\vec{J}\times\vec{B}}{cn^-} + \frac{1}{en^-}\vec{\nabla}\cdot\vec{P}'^- - \frac{1}{en^-}p_{+-}. \qquad (47)$$

The real difficulties are now concentrated on the last two terms on the right-hand side of Eq. (47). The term involving p_{+-}, if the deformations of the distribution function from local Maxwellian are small, should be proportional to the relative velocity of the two types of particles. We shall take this term equal to $-\eta J$ where η is defined by

$$\eta = |p_{+-}|/en^-|J|. \qquad (48)$$

Actually η is frequency and magnetic field dependent, and not even scalar, effects which will be discussed in other papers.

The term involving the stress tensor is troublesome. We may take this scalar and isotropic only in a collision dominated theory, where there are many collisions during a characteristic time, in which case

$$\vec{P} = p\vec{I} \qquad (49)$$

and

$$\frac{d}{dt}(p\rho^{-5/3}) = 0. \qquad (50)$$

For rapid changes in which the internal kinetic energy changes in only one or two directions the appropriate γ is 3 or 2, respectively. Actually in the presence of thermal gradients heat flow terms appear in Eq. (47) but this point will not be discussed further at this time.

The equation of motion under these approximations is

$$\rho\frac{d\vec{U}_0}{dt} = \frac{\vec{J}}{c}\times\vec{B} - \vec{\nabla}\cdot\vec{P}. \qquad (51)$$

(The charge neutrality condition $n_- \approx Zn_+$ has been invoked to throw out the term $\sigma\vec{E}$ that should appear in Eq. (51).)

Let us now try to estimate the size of the terms appearing in the generalized Ohm's law, to see under what situation certain terms or another might be omitted from these equations.

If we let L and T measure typical spatial and temporal variations, then

$$\frac{4\pi\vec{J}}{\omega_p T} \approx \vec{E} + \frac{\vec{U}_0}{c}\times\vec{B} + \frac{\Theta}{eL}\vec{1}_L - \frac{\vec{J}\times\vec{B}}{cn^-} - \frac{4\pi\nu_c}{\omega_p^2}\vec{J}. \qquad (52)$$

We can simplify this equation further by eliminating the term involving $\vec{J}\times\vec{B}$ in Eq. (47) via the equation of motion. This yields

$$\frac{4\pi\vec{J}}{\omega_{pe}^2 T} \approx \vec{E} + \frac{\vec{U}_0}{c}\times\vec{B} - \frac{\Theta}{eL}\vec{1}_L - \frac{m^+}{e}\frac{\vec{U}_0}{T} - \frac{4\pi\nu_c}{\omega_p^2}\vec{J}. \qquad (53)$$

In terms of characteristic frequencies and speeds this becomes

$$\frac{4\pi\vec{J}}{\omega_{pe}^2 T} \approx \vec{E} + \frac{\vec{U}_0}{c}\times\vec{B} + \frac{c_s a_{ci}}{cL}\vec{B} - \frac{\vec{U}_0}{c}\times\frac{\vec{B}}{\omega_{ci}T} - \frac{4\pi\nu_c}{\omega_p^2}\vec{J}, \qquad (54)$$

where

$$c_s = \text{sound speed} \sim (\Theta/m^+)^{1/2}, \qquad (55)$$

$$\omega_{ci} = \text{ion gyro-frequency} = eB/m^+c, \qquad (56)$$

$$a_{ci} = \text{ion gyro-radius} = (\Theta/m^+)^{1/2}/\omega_{ci}. \qquad (57)$$

In Eq. (54) we have performed a very dangerous simplification in annihilating the vectorial nature of the equation. It so often happens, especially in the presence of a strong magnetic field, that terms which are large in one direction may vanish in another, so that for every particular problem one must respect the vectorial character in performing estimates on the size of the various terms.

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