entropic

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2-9 The damped harmonic oscillator. The equation of motion for a particle subject to a linear restoring force and a frictional force proportional to its velocity is [Eq. (2-85)]

$$m\ddot{x} + b\dot{x} + kx = 0, \tag{2-124}$$

where the dots stand for time derivatives. Applying the method of Section 2-8, we make the substitution (2-98) and obtain

$$mp^2 + bp + k = 0. \tag{2-125}$$

The solution is

$$p = -\frac{b}{2m} \pm \left[\left(\frac{b}{2m}\right)^2 - \frac{k}{m}\right]^{1/2}. \tag{2-126}$$

We distinguish three cases: (a) $\boxed{k/m > (b/2m)^2 \quad \omega_0^2 > \gamma^2}$ (b) $k/m < (b/2m)^2$, and (c) $k/m = (b/2m)^2$.

In case (a), we make the substitutions

under-damped

$$\omega_0 \equiv \sqrt{\frac{k}{m}}, \tag{2-127}$$

$$\gamma \equiv \frac{b}{2m}, \tag{2-128}$$

$$\omega_1 \equiv (\overbrace{\omega_0^2 - \gamma^2}^{\Delta(\omega^2)})^{1/2}, \tag{2-129}$$

where γ is called the damping coefficient and $(\omega_0/2\pi)$ is the natural frequency of the undamped oscillator. There are now two solutions for p:

$$p = -\gamma \pm i\omega_1. \tag{2-130}$$

The general solution of the differential equation is therefore

$$x = C_1 e^{-\gamma t + i\omega_1 t} + C_2 e^{-\gamma t - i\omega_1 t}. \tag{2-131}$$

Setting

$$C_1 = \tfrac{1}{2}Ae^{i\theta}, \qquad C_2 = \tfrac{1}{2}Ae^{-i\theta}, \tag{2-132}$$

we have

$$\boxed{x = Ae^{-\gamma t}\cos(\omega_1 t + \theta).} \tag{2-133}$$

This corresponds to an oscillation of frequency $(\omega_1/2\pi)$ with an amplitude $Ae^{-\gamma t}$ which decreases exponentially with time (Fig. 2-4). The constants A and θ depend upon the initial conditions. The frequency of oscillation is less than without damping. The solution (2-133) can also be written

$$x = e^{-\gamma t}(B_1 \cos\omega_1 t + B_2 \sin\omega_1 t). \tag{2-134}$$

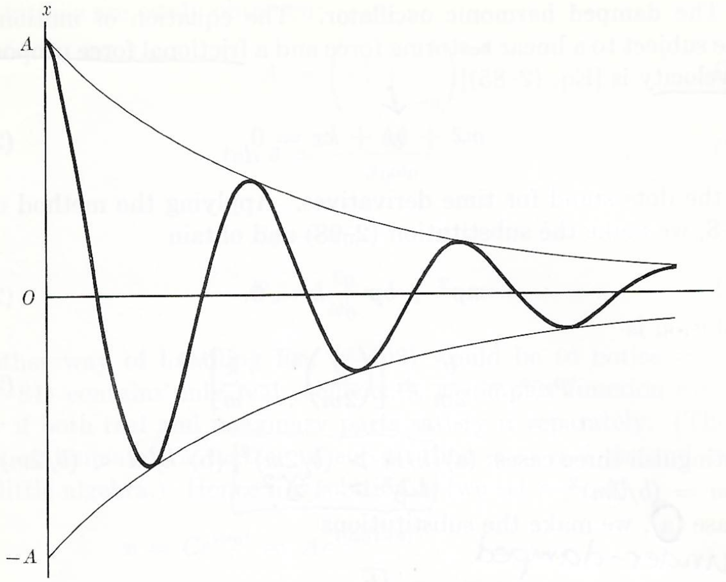

FIG. 2–4. Motion of damped harmonic oscillator. Heavy curve: $x = Ae^{-\gamma t} \cos \omega t$, $\gamma = \omega/8$. Light curve: $x = \pm Ae^{-\gamma t}$.

In terms of the constants ω_0 and γ, Eq. (2–124) can be written

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0. \tag{2–135}$$

This form of the equation is often used in discussing mechanical oscillations.

The total energy of the oscillator is

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2. \tag{2–136}$$

approx

In the important case of small damping, $\gamma \ll \omega_0$, we can set $\omega_1 \doteq \omega_0$ and neglect γ compared with ω_0, and we have for the energy corresponding to the solution (2–133), approximately,

$$E \doteq \tfrac{1}{2}kA^2e^{-2\gamma t} = E_0e^{-2\gamma t}. \tag{2–137}$$

Thus the energy falls off exponentially at twice the rate at which the amplitude decays. The fractional rate of decline or *logarithmic derivative* of E is

$$\frac{1}{E}\frac{dE}{dt} = \frac{d \ln E}{dt} = -2\gamma. \tag{2–138}$$

We now consider case (b), $(\omega_0 < \gamma)$. In this case, the two solutions for p are

over-damped

$(\Delta\omega)^2$

$$p = -\gamma_1 = -\gamma - (\gamma^2 - \omega_0^2)^{1/2},$$
$$p = -\gamma_2 = -\gamma + (\gamma^2 - \omega_0^2)^{1/2}. \tag{2–139}$$

The general solution is

$$x = C_1 e^{-\gamma_1 t} + C_2 e^{-\gamma_2 t}. \tag{2–140}$$

These two terms both decline exponentially with time, one at a faster rate than the other. The constants C_1 and C_2 may be chosen to fit the initial conditions. The reader should determine them for two important cases: $x_0 \neq 0$, $v_0 = 0$ and $x_0 = 0$, $v_0 \neq 0$, and draw curves $x(t)$ for the two cases.

In case (c), $(\omega_0 = \gamma)$, we have only one solution for p:

critical damping

$$p = -\gamma. \tag{2–141}$$

The corresponding solution for x is

$$x = e^{-\gamma t}. \tag{2–142}$$

We now show that, in this case, another solution is

$$x = te^{-\gamma t}. \tag{2–143}$$

To prove this, we compute

$$\dot{x} = e^{-\gamma t} - \gamma t e^{-\gamma t},$$
$$\ddot{x} = -2\gamma e^{-\gamma t} + \gamma^2 t e^{-\gamma t}. \tag{2–144}$$

The left side of Eq. (2–135) is, for this x,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = (\omega_0^2 - \gamma^2) t e^{-\gamma t}. \tag{2–145}$$

This is zero if $\omega_0 = \gamma$. Hence the general solution in case $\omega_0 = \gamma$ is

$$x = (C_1 + C_2 t) e^{-\gamma t}. \tag{2–146}$$

This function declines exponentially with time at a rate intermediate between that of the two exponential terms in Eq. (2–140):

$$\gamma_1 > \gamma > \gamma_2. \tag{2–147}$$

Hence the solution (2–146) falls to zero faster after a sufficiently long time than the solution (2–140), except in the case $C_2 = 0$ in Eq. (2–140). Cases (a), (b), and (c) are important in problems involving mechanisms which approach an equilibrium position under the action of a frictional damping

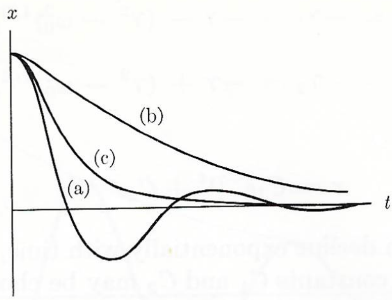

FIG. 2-5. Return of harmonic oscillator to equilibrium. (a) Underdamped. (b) Overdamped. (c) Critically damped.

force, e.g., pointer reading meters, hydraulic and pneumatic spring returns for doors, etc. In most cases, it is desired that the mechanism move quickly and smoothly to its equilibrium position. For a given damping coefficient γ, or for a given ω_0, this is accomplished in the shortest time without overshoot if $\omega_0 = \gamma$ [case (c)]. This case is called *critical damping.* If $\omega_0 < \gamma$, the system is said to be *overdamped;* it behaves sluggishly and does not return as quickly to $x = 0$ as for critical damping. If $\omega_0 > \gamma$, the system is said to be *underdamped;* the coordinate x then overshoots the value $x = 0$ and oscillates. Note that at critical damping, $\omega_1 = 0$, so that the period of oscillation becomes infinite. The behavior is shown in Fig. 2-5 for the case of a system displaced from equilibrium and released ($x_0 \neq 0, v_0 = 0$). The reader should draw similar curves for the case where the system is given a sharp blow at $t = 0$ (i.e., $x_0 = 0, v_0 \neq 0$).

+damped in Steady-State

2-10 The forced harmonic oscillator. The harmonic oscillator subject to an external applied force is governed by Eq. (2-86). In order to simplify the problem of solving this equation, we state the following theorem:

!!

THEOREM III. *If $x_i(t)$ is a solution of an inhomogeneous linear equation [e.g., Eq. (2-86)], and $x_h(t)$ is a solution of the corresponding homogeneous equation [e.g., Eq. (2-85)], then $x(t) = x_i(t) + x_h(t)$ is also a solution of the inhomogeneous equation.*

This theorem applies whether the coefficients in the equation are constants or functions of t. The proof is a matter of straightforward substitution, and is left to the reader. In consequence of Theorem III, if we know the general solution x_h of the homogeneous equation (2-85) (we found this in Section 2-9), then we need find only one particular solution x_i of the inhomogeneous equation (2-86). For we can add x_i to x_h and obtain a solution of Eq. (2-86) which contains two arbitrary constants and is therefore the general solution.

The most important case is that of a sinusoidally oscillating applied force. If the applied force oscillates with angular frequency ω and amplitude F_0, the equation of motion is

$$m\,\frac{d^2x}{dt^2} + b\,\frac{dx}{dt} + kx = F_0 \cos(\omega t + \theta_0), \tag{2–148}$$

where θ_0 is a constant specifying the phase of the applied force. There are, of course, many solutions of Eq. (2–148), of which we need find only one. From physical considerations, we expect that one solution will be a steady oscillation of the coordinate x at the same frequency as the applied force:

$$x = A_s \cos(\omega t + \theta_s). \tag{2–149}$$

The amplitude A_s and phase θ_s of the oscillations in x will have to be determined by substituting Eq. (2–149) in Eq. (2–148). This procedure is straightforward and leads to the correct answer. The algebra is simpler, however, if we write the force as the real part of a complex function:*

$$F(t) = Re(\mathrm{F}_0 e^{i\omega t}), \tag{2–150}$$

$$\mathrm{F}_0 = F_0 e^{i\theta_0}. \tag{2–151}$$

Thus if we can find a solution $\mathrm{x}(t)$ of

$$m\,\frac{d^2\mathrm{x}}{dt^2} + b\,\frac{d\mathrm{x}}{dt} + k\mathrm{x} = \mathrm{F}_0 e^{i\omega t}, \tag{2–152}$$

then, by splitting the equation into real and imaginary parts, we can show that the real part of $\mathrm{x}(t)$ will satisfy Eq. (2–148). We assume a solution of the form

$$\mathrm{x} = \mathrm{x}_0 e^{i\omega t},$$

so that

$$\dot{\mathrm{x}} = i\omega \mathrm{x}_0 e^{i\omega t}, \qquad \ddot{\mathrm{x}} = -\omega^2 \mathrm{x}_0 e^{i\omega t}. \tag{2–153}$$

Substituting in Eq. (2–152), we solve for x_0:

$$\mathrm{x}_0 = \frac{\mathrm{F}_0/m}{\omega_0^2 - \omega^2 + 2i\gamma\omega}. \tag{2–154}$$

The solution of Eq. (2–152) is therefore

$$\mathrm{x} = \mathrm{x}_0 e^{i\omega t} = \frac{(\mathrm{F}_0/m)e^{i\omega t}}{\omega_0^2 - \omega^2 + 2i\gamma\omega}. \tag{2–155}$$

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* Note the use of roman type (F, x) to distinguish complex quantities from the corresponding real quantities (F, x).

We are often more interested in the velocity

$$\dot{\text{x}} = \frac{i\omega \text{F}_0}{m} \frac{e^{i\omega t}}{\omega_0^2 - \omega^2 + 2i\gamma\omega}. \tag{2-156}$$

The simplest way to write Eq. (2–156) is to express all complex factors in polar form [Eq. (2–109)]:

$$i = e^{i\pi/2}, \tag{2-157}$$

$$\omega_0^2 - \omega^2 + 2i\gamma\omega = [(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{1/2} \exp\left(i \tan^{-1} \frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right). \tag{2-158}$$

If we use these expressions, Eq. (2–156) becomes

$$\dot{\text{x}} = \frac{\omega F_0}{m[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{1/2}} e^{i(\omega t + \theta_0 + \beta)}, \tag{2-159}$$

where

$$\beta = \frac{\pi}{2} - \tan^{-1} \frac{2\gamma\omega}{\omega_0^2 - \omega^2} = \tan^{-1} \frac{\omega_0^2 - \omega^2}{2\gamma\omega}, \tag{2-160}$$

$$\sin \beta = \frac{\omega_0^2 - \omega^2}{[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{1/2}}, \tag{2-161}$$

$$\cos \beta = \frac{2\gamma\omega}{[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{1/2}}. \tag{2-162}$$

By Eq. (2–159),

$$\begin{aligned} \dot{x} &= Re(\dot{\text{x}}) \\ &= \frac{F_0}{m} \frac{\omega}{[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{1/2}} \cos(\omega t + \theta_0 + \beta), \end{aligned} \tag{2-163}$$

and

$$\begin{aligned} x &= Re(\text{x}) = Re\,(\dot{\text{x}}/i\omega) \\ &= \frac{F_0}{m} \frac{1}{[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{1/2}} \sin(\omega t + \theta_0 + \beta). \end{aligned} \tag{2-164}$$

This is a particular solution of Eq. (2–148) containing no arbitrary constants. By Theorem III and Eq. (2–133), the general solution (for the underdamped oscillator) is

$$x = Ae^{-\gamma t} \cos(\omega_1 t + \theta) + \frac{F_0/m}{[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{1/2}} \sin(\omega t + \theta_0 + \beta). \tag{2-165}$$

max amp at $\omega^2 = \omega_0^2 - 2\gamma^2$ ✓

This solution contains two arbitrary constants A, θ, whose values are determined by the initial values x_0, v_0 at $t = 0$. The first term dies out exponentially in time and is called the *transient*. The second term is called the *steady state*, and oscillates with constant amplitude. The transient depends on the initial conditions. The steady state which remains after the transient dies away is independent of the initial conditions.

In the steady state, the rate at which work is done on the oscillator by the applied force is

$$\dot{x}F(t) = \frac{F_0^2}{m} \frac{\omega}{[(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2]^{1/2}} \cos(\omega t + \theta_0) \cos(\omega t + \theta_0 + \beta)$$

$$= \frac{F_0^2}{m} \frac{\omega \cos\beta \cos^2(\omega t + \theta_0)}{[(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2]^{1/2}} - \frac{F_0^2}{2m} \frac{\omega \sin\beta \sin 2(\omega t + \theta_0)}{[(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2]^{1/2}}. \tag{2–166}$$

The last term on the right is zero on the average, while the average value of $\cos^2(\omega t + \theta_0)$ over a complete cycle is $\frac{1}{2}$. Hence the average power delivered by the applied force is

$$P_{\text{av}} = \langle \dot{x}F(t) \rangle_{\text{av}} = \frac{F_0^2 \cos\beta}{2m} \frac{\omega}{[(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2]^{1/2}}, \tag{2–167}$$

or

$$P_{\text{av}} = \tfrac{1}{2} F_0 \dot{x}_m \cos\beta, \tag{2–168}$$

where $\dot{x}_m$ is the maximum value of $\dot{x}$. A similar relation holds for power delivered to an electrical circuit. The factor $\cos\beta$ is called the *power factor*. In the electrical case, β is the phase angle between the current and the applied emf. Using formula (2–162) for $\cos\beta$, we can rewrite Eq. (2–167):

peaks @ $\omega = \omega_0$

$$P_{\text{av}} = \frac{F_0^2}{m} \frac{\gamma\omega^2}{(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2}. \rightsquigarrow \frac{1}{\omega^2} \tag{2–169}$$

It is easy to show that in the steady state power is supplied to the oscillator at the same average rate that power is being dissipated by friction, as of course it must be. The power P_{av} has a maximum for $\omega = \omega_0$. In Fig. 2–6, the power P_{av} (in arbitrary units) and the phase of β of steady-state forced oscillations are plotted against ω for two values of γ. The heavy curves are for small damping; the light curves are for greater damping. Formula (2–169) can be simplified somewhat in case $\gamma \ll \omega_0$. In this case, P_{av} is large only near the resonant frequency ω_0, and we shall deduce a formula valid near $\omega = \omega_0$. Defining

$$\Delta\omega = \omega - \omega_0, \tag{2–170}$$

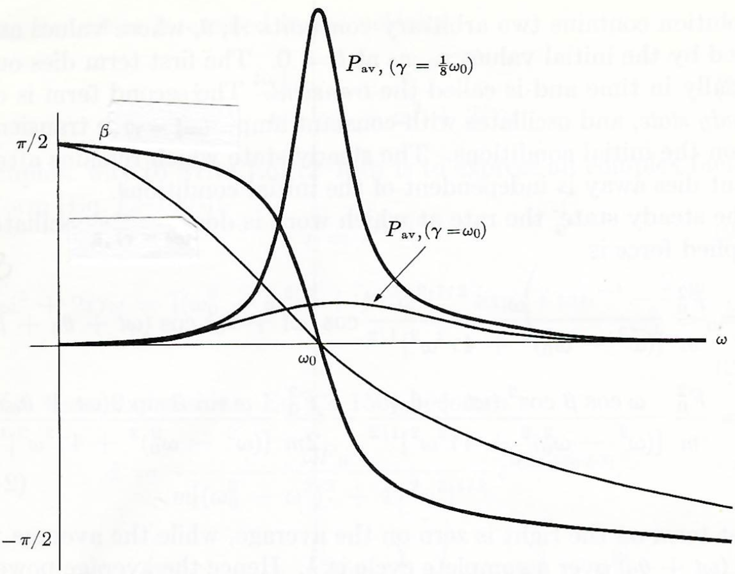

FIG. 2–6. Power and phase of forced harmonic oscillations.

and assuming $\Delta\omega \ll \omega_0$, we have

$$(\omega^2 - \omega_0^2) = (\omega + \omega_0)\,\Delta\omega \doteq 2\omega_0\,\Delta\omega, \tag{2–171}$$

$$\omega^2 \doteq \omega_0^2. \tag{2–172}$$

Hence

$$P_{av} \doteq \frac{F_0^2}{4m}\,\frac{\gamma}{(\Delta\omega)^2 + \gamma^2}. \tag{2–173}$$

This simple formula gives a good approximation to P_{av} near resonance. The corresponding formula for β is

$$\cos\beta \doteq \frac{\gamma}{[(\Delta\omega)^2 + \gamma^2]^{1/2}}, \qquad \sin\beta \doteq \frac{-\Delta\omega}{[(\Delta\omega)^2 + \gamma^2]^{1/2}}. \tag{2–174}$$

When $\omega \ll \omega_0$, $\beta \doteq \pi/2$, and Eq. (2–164) becomes

$$x \doteq \frac{F_0}{\omega_0^2 m}\cos(\omega t + \theta_0) = \frac{F(t)}{k}. \tag{2–175}$$

This result is easily interpreted physically; when the force varies slowly, the particle moves in such a way that the applied force is just balanced by the restoring force. When $\omega \gg \omega_0$, $\beta \doteq -\pi/2$, and Eq. (2–164) becomes

$$x \doteq -\frac{F_0}{\omega^2 m}\cos(\omega t + \theta_0) = -\frac{F(t)}{\omega^2 m}. \tag{2–176}$$

The motion now depends only on the mass of the particle and on the frequency of the applied force, and is independent of the friction and the restoring force. This result is, in fact, identical with that obtained in Section 2–3 [see Eqs. (2–15) and (2–19)] for a free particle subject to an oscillating force.

We can apply the result (2–165) to the case of an electron bound to an equilibrium position $x = 0$ by an elastic restoring force, and subject to an oscillating electric field:

$$E_x = E_0 \cos \omega t, \tag{2–177}$$

$$F = -eE_0 \cos \omega t. \tag{2–178}$$

The motion will be given by

$$x = Ae^{-\gamma t} \cos (\omega_1 t + \theta) - \frac{eE_0}{m} \frac{\sin (\omega t + \beta)}{[(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2]^{1/2}}. \tag{2–179}$$

The term of interest here is the second one, which is independent of the initial conditions and oscillates with the frequency of the electric field. Expanding the second term, we get

$$\begin{aligned} x &= -\frac{eE_0}{m} \frac{\sin \beta \cos \omega t}{[(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2]^{1/2}} - \frac{eE_0}{m} \frac{\cos \beta \sin \omega t}{[(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2]^{1/2}} \\ &= \frac{-eE_0 \cos \omega t}{m} \frac{\omega_0^2 - \omega^2}{[(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2]} \qquad \text{Polarization} \\ &\quad - \frac{eE_0 \sin \omega t}{m} \frac{2\gamma\omega}{[(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2]}. \qquad \text{Absorption of Energy} \end{aligned} \tag{2–180}$$

The first term represents an oscillation of x in phase with the applied force at low frequencies, 180° out of phase at high frequencies. The second term represents an oscillation of x that is 90° out of phase with the applied force, the velocity $\dot{x}$ for this term being in phase with the applied force. Hence the second term corresponds to an absorption of energy from the applied force. The second term contains a factor γ and is therefore small, if $\gamma \ll \omega_0$, except near resonance. If we imagine a dielectric medium consisting of electrons bound by elastic forces to positions of equilibrium, then the first term in Eq. (2–180) will represent an electric polarization proportional to the applied oscillating electric field, while the second term will represent an absorption of energy from the electric field. Near the resonant frequency, the dielectric medium will absorb energy, and will be opaque to electromagnetic radiation. Above the resonant frequency, the displacement of the electrons is out of phase with the applied force, and the resulting electric polarization will be out of phase with the applied electric field. The dielectric constant and index of refraction will be less than one. For very high frequencies, the first term of Eq. (2–180) approaches the last term of Eq. (2–18), and the electrons behave as if they were free. Below the resonant frequency, the electric polarization will be in phase with the applied electric field, and the dielectric constant and index of refraction will be greater than one.

Computing the dielectric constant from the first term in Eq. (2–180), in the same manner as for a free electron [see Eqs. (2–20)–(2–26)], we find, for N electrons per unit volume:

$$\epsilon = 1 + \frac{4\pi N e^2}{m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}. \tag{2–181}$$

The index of refraction for electromagnetic waves ($\mu = 1$) is

$$n = \frac{c}{v} = (\mu\epsilon)^{1/2} = \epsilon^{1/2}. \tag{2–182}$$

For very high or very low frequencies, Eq. (2–181) becomes

$$\epsilon \doteq 1 + \frac{4\pi N e^2}{m\omega_0^2}, \quad \omega \ll \omega_0, \tag{2–183}$$

$$\epsilon \doteq 1 - \frac{4\pi N e^2}{m\omega^2}, \quad \omega \gg \omega_0. \tag{2–184}$$

The mean rate of energy absorption per unit volume is given by Eq. (2–169):

$$\frac{dE}{dt} = \frac{N e^2 E_0^2}{m} \frac{\gamma\omega^2}{(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2}. \tag{2–185}$$

The resulting dielectric constant and energy absorption versus frequency are plotted in Fig. 2–7. Thus the dielectric constant is constant and greater than one at low frequencies, increases as we approach the resonant frequency, falls to less than one in the region of "anomalous dispersion" where there is strong absorption of electromagnetic radiation, and then rises, approaching one at high frequencies. The index of refraction will follow a similar curve. This is precisely the sort of behavior which is exhibited by matter in all forms. Glass, for example, has a constant dielectric constant at low frequencies; in the region of visible light its index of refraction increases with frequency; and it becomes opaque in a certain

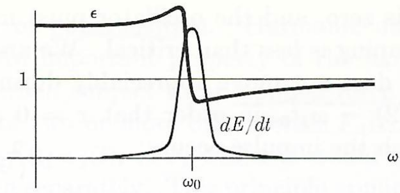

FIG. 2–7. Dielectric constant and energy absorption for medium containing harmonic oscillators.

band in the ultraviolet. X-rays are transmitted with an index of refraction very slightly less than one. A more realistic model of a transmitting medium would result from assuming several different resonant frequencies corresponding to electrons bound with various values of the spring constant k. This picture is then capable of explaining most of the features in the experimental curves for ϵ or n vs. frequency. Not only is there qualitative agreement, but the formulas (2–181)–(2–185) agree quantitatively with experimental results, provided the constants N, ω_0, and γ are properly chosen for each material. The success of this theory was one of the reasons for the adoption, until the year 1913, of the "jelly model" of the atom, in which electrons were imagined embedded in a positively charged jelly in which they oscillated as harmonic oscillators. The experiments of Rutherford in 1913 forced physicists to adopt the "planetary" model of the atom, but this model was unable to explain even qualitatively the optical and electromagnetic properties of matter until the advent of quantum mechanics. The result of the quantum-mechanical treatment is that, for the interaction of matter and radiation, the simple oscillator picture gives essentially correct results when the constants are properly chosen.*

We now consider an applied force $F(t)$ which is large only during a short time interval δt and is zero or negligible at all other times. Such a force is called an impulse, and corresponds to a sudden blow. We assume the oscillator initially at rest at $x = 0$, and we assume the time δt so short that the mass moves only a negligibly small distance while the force is acting. According to Eq. (2–4), the momentum just after the force is applied will equal the impulse delivered by the force:

$$mv_0 = p_0 = \int F\,dt, \tag{2–186}$$

where v_0 is the velocity just after the impulse, and the integral is taken over the time interval δt during which the force acts. After the impulse,

* See John C. Slater, *Quantum Theory of Matter*. New York: McGraw-Hill Book Co., 1951. (Page 378.)

the applied force is zero, and the oscillator must move according to Eq. (2–133) if the damping is less than critical. We are assuming δt so small that the oscillator does not move appreciably during this time, hence we choose $\theta = -(\pi/2) - \omega_1 t_0$, in order that $x = 0$ at $t = t_0$, where t_0 is the instant at which the impulse occurs:

$\omega_1^2 \equiv \omega_0^2 - \gamma^2$

$$x = Ae^{-\gamma t} \sin [\omega_1(t - t_0)]. \tag{2–187}$$

The velocity at $t = t_0$ is

$$v_0 = \omega_1 A e^{-\gamma t_0}. \tag{2–188}$$

Thus

$$A = \frac{v_0}{\omega_1} e^{\gamma t_0}. \tag{2–189}$$

The solution when an impulse p_0 is delivered at $t = t_0$ to an oscillator at rest is therefore

$$x = \begin{cases} 0, & t \leq t_0, \\ \dfrac{p_0}{m\omega_1} e^{-\gamma(t-t_0)} \sin [\omega_1(t - t_0)], & t > t_0. \end{cases} \tag{2–190}$$

Here we have neglected the short time δt during which the force acts.

We see that the result of an impulse-type force depends only on the total impulse p_0 delivered, and is independent of the particular form of the function $F(t)$, provided only that $F(t)$ is negligible except during a very short time interval δt. Several possible forms of $F(t)$ which have this property are listed below:

$$F(t) = \begin{cases} 0, & t < t_0, \\ p_0/\delta t, & t_0 \leq t \leq t_0 + \delta t, \\ 0, & t > t_0 + \delta t, \end{cases} \tag{2–191}$$

$$F(t) = \frac{p_0\, \delta t}{\pi} \frac{1}{(t - t_0)^2 + (\delta t)^2}, \qquad -\infty < t < \infty, \tag{2–192}$$

$$F(t) = \frac{p_0}{\delta t\sqrt{\pi}} \exp\left[-\frac{(t - t_0)^2}{(\delta t)^2}\right], \qquad -\infty < t < \infty. \tag{2–193}$$

The reader may verify that each of these functions is negligible except within an interval of the order of δt around t_0, and that the total impulse delivered by each is p_0. The exact solution of Eq. (2–86) with $F(t)$ given by any of the above expressions must reduce to Eq. (2–190) when $\delta t \to 0$ (see Problem 23).