Symon, Mechanics (1960)

There are many cases, however, in which a group of particles forms a system which seems to have some identity of its own independent of other particles and systems of particles. An atomic nucleus, made up of neutrons and protons, is an example, as is an atom, made up of nucleus and electrons, or a molecule, composed of nuclei and electrons, or the collection of particles which make up a baseball. In all such cases, it turns out that the internal forces are much stronger than the external ones, and the acceleration $\ddot{\mathbf{R}}$ is small, so that the internal equations of motion (4–131) depend essentially only on the internal forces, and their solutions represent internal motions which are nearly independent of the external forces and of the motion of the system as a whole. The system viewed externally then behaves like a single particle with coordinate vector $\mathbf{R}$, mass M, acted on by the (external) force $\mathbf{F}$, but a particle which has, in addition to its "orbital" energy, momentum, and angular momentum associated with the motion of its center of mass, an intrinsic or internal energy and angular momentum associated with its internal motion. The orbital and intrinsic parts of the energy, momentum, and angular momentum can be identified in Eqs. (4–127), (4–128), and (4–129). The internal angular momentum is usually called *spin* and is independent of the position or velocity of the center of mass relative to the origin about which the total angular momentum is to be computed. So long as the external forces are small, this approximate representation of the system as a single particle is valid. Whenever the external forces are strong enough to affect appreciably the internal motion, the separation into problems of internal and of orbital motions breaks down and the system begins to lose its individuality. Some of the central problems at the frontiers of present-day physical theories are concerned with bridging the gap between a loose collection of particles and a system with sufficient individuality to be treated as a single particle.

[Collisions]

$\gamma = 0$

4–10 Two coupled harmonic oscillators. A very commonly occurring type of mechanical system is one in which several harmonic oscillators interact with one another. As a typical example of such a system, consider the mechanical system shown in Fig. 4–10, consisting of two masses m_1, m_2 fastened to fixed supports by springs whose elastic constants are k_1, k_2, and connected by a third spring of elastic constant k_3. We suppose the masses are free to move only along the x-axis; they may, for example, slide along a rail. If spring k_3 were not present, the two masses would

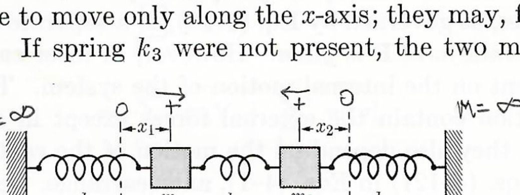

FIG. 4–10. A simple model of two coupled harmonic oscillators.

vibrate independently in simple harmonic motion with angular frequencies (neglecting damping)

$$\omega_{10}^0 = \sqrt{\frac{k_1}{m_1}}, \qquad \omega_{20}^0 = \sqrt{\frac{k_2}{m_2}}. \tag{4–132}$$

We wish to investigate the effect of coupling these two oscillators together by means of the spring k_3. We describe the positions of the two masses by specifying the distances x_1 and x_2 that the springs k_1 and k_2 have been stretched from their equilibrium positions. We assume for simplicity that when springs k_1 and k_2 are relaxed ($x_1 = x_2 = 0$), spring k_3 is also relaxed. The amount by which spring k_3 is compressed is then $(x_1 + x_2)$. The equations of motion for the masses m_1, m_2 (neglecting friction) are

$$m_1\ddot{x}_1 = -k_1x_1 - k_3(x_1 + x_2), \tag{4–133}$$

$$m_2\ddot{x}_2 = -k_2x_2 - k_3(x_1 + x_2). \tag{4–134}$$

We rewrite these in the form

$$m_1\ddot{x}_1 + k_1'x_1 + k_3x_2 = 0, \tag{4–135}$$

$$m_2\ddot{x}_2 + k_2'x_2 + k_3x_1 = 0, \tag{4–136}$$

where

$$k_1' = k_1 + k_3, \tag{4–137}$$

$$k_2' = k_2 + k_3. \tag{4–138}$$

We have two second-order linear differential equations to solve simultaneously. If the third terms were not present, the equations would be independent of one another, and we would have independent harmonic vibrations of x_1 and x_2 at frequencies

$$\omega_{10} = \sqrt{\frac{k_1'}{m_1}}, \tag{4–139}$$

$$\omega_{20} = \sqrt{\frac{k_2'}{m_2}}. \tag{4–140}$$

These are the frequencies with which each mass would vibrate if the other were held fixed. Thus the first effect of the coupling spring is simply to change the frequency of independent vibration of each mass, due to the fact that each mass is now pushed ~~held in position~~ by two springs instead of one. The third terms in Eqs. (4–135) and (4–136) give rise to a coupling between the motions of the two masses, so that they no longer move independently.

$\gamma = 0$

We may solve Eqs. (4–135), (4–136) by an extension of the method of Section 2–8 applicable to any set of simultaneous linear differential equations with constant coefficients. We assume that

$$x_1 = C_1 e^{pt}, \tag{4–141}$$

$$x_2 = C_2 e^{pt}, \tag{4–142}$$

phase-locked

where C_1, C_2 are constants. Note that the same time dependence is assumed for both x_1 and x_2, in order that the factor e^{pt} will cancel out when we substitute in Eqs. (4–135) and (4–136):

$$(m_1 p^2 + k_1')C_1 + k_3 C_2 = 0, \tag{4–143}$$

$$(m_2 p^2 + k_2')C_2 + k_3 C_1 = 0. \tag{4–144}$$

We now have two algebraic equations in the three unknown quantities C_1, C_2, p. We note that either Eq. (4–143) or (4–144) can be solved for the ratio C_2/C_1:

$$\frac{C_2}{C_1} = -\frac{m_1 p^2 + k_1'}{k_3} = -\frac{k_3}{m_2 p^2 + k_2'}. \tag{4–145}$$

The two values of C_2/C_1 must be equal, and we have an equation for p:

$$\frac{m_1 p^2 + k_1'}{k_3} = \frac{k_3}{m_2 p^2 + k_2'}, \tag{4–146}$$

which may be rearranged as a quadratic equation in p^2, called the *secular equation:*

$$m_1 m_2 p^4 + (m_2 k_1' + m_1 k_2')p^2 + (k_1' k_2' - k_3^2) = 0, \tag{4–147}$$

whose solutions are

$$\begin{aligned} p^2 &= -\frac{1}{2}\left(\frac{k_1'}{m_1} + \frac{k_2'}{m_2}\right) \pm \left[\frac{1}{4}\left(\frac{k_1'}{m_1} + \frac{k_2'}{m_2}\right)^2 - \frac{k_1' k_2'}{m_1 m_2} + \frac{k_3^2}{m_1 m_2}\right]^{1/2} \\ &= -\frac{1}{2}(\omega_{10}^2 + \omega_{20}^2) \pm \left[\frac{1}{4}(\omega_{10}^2 - \omega_{20}^2)^2 + \frac{k_3^2}{m_1 m_2}\right]^{1/2}. \end{aligned} \tag{4–148}$$

It is not hard to show that the quantity in brackets is less than the square of the first term, so that we have two negative solutions for p^2. If we assume that $\omega_{10} \geq \omega_{20}$, the solutions for p^2 are

$$\begin{aligned} p^2 &= -\omega_1^2 = -(\omega_{10}^2 + \tfrac{1}{2}\Delta\omega^2), \\ p^2 &= -\omega_2^2 = -(\omega_{20}^2 - \tfrac{1}{2}\Delta\omega^2), \end{aligned} \tag{4–149}$$

where

$$\Delta\omega^2 \equiv (\omega_{10}^2 - \omega_{20}^2)\left[\left(1 + \frac{4\kappa^4}{(\omega_{10}^2 - \omega_{20}^2)^2}\right)^{1/2} - 1\right], \tag{4-150}$$

with the abbreviation

$$\kappa^2 \equiv \frac{k_3}{\sqrt{m_1 m_2}}, \tag{4-151}$$

where κ is the coupling constant. $\boxed{\text{If } \omega_{10} = \omega_{20},}$ Eq. (4-150) reduces to at crossing

$$\Delta\omega^2 = 2\kappa^2. \sim 2k_3^2 \tag{4-152}$$

The four solutions for p are

$$p = \pm i\omega_1, \qquad \pm i\omega_2. \tag{4-153}$$

If $p^2 = -\omega_1^2$, Eq. (4-145) can be written

$$\frac{C_2}{C_1} = \frac{m_1}{k_3}(\omega_1^2 - \omega_{10}^2) = \frac{\Delta\omega^2}{2\kappa^2}\sqrt{\frac{m_1}{m_2}}, \tag{4-154}$$

and if $p^2 = -\omega_2^2$, it can be written

inversion !!

$$\frac{C_1}{C_2} = \frac{m_2}{k_3}(\omega_2^2 - \omega_{20}^2) = -\frac{\Delta\omega^2}{2\kappa^2}\sqrt{\frac{m_2}{m_1}}. \tag{4-155}$$

By substituting from Eq. (4-153) in Eqs. (4-141), (4-142), we get four solutions of Eqs. (4-135) and (4-136), provided the ratio C_2/C_1 is chosen according to Eq. (4-154) or (4-155). Each of these solutions involves one arbitrary constant (C_1 or C_2). Since the equations (4-135), (4-136) are linear, the sum of these four solutions will also be a solution, and is in fact the general solution, for it will contain four arbitrary constants (say C_1, C_1', C_2, C_2'):

$$x_1 = C_1 e^{i\omega_1 t} + C_1' e^{-i\omega_1 t} - \frac{\Delta\omega^2}{2\kappa^2}\sqrt{\frac{m_2}{m_1}}\, C_2 e^{i\omega_2 t} - \frac{\Delta\omega^2}{2\kappa^2}\sqrt{\frac{m_2}{m_1}}\, C_2' e^{-i\omega_2 t}, \tag{4-156}$$

$$x_2 = \frac{\Delta\omega^2}{2\kappa^2}\sqrt{\frac{m_1}{m_2}}\, C_1 e^{i\omega_1 t} + \frac{\Delta\omega^2}{2\kappa^2}\sqrt{\frac{m_1}{m_2}}\, C_1' e^{-i\omega_1 t} + C_2 e^{i\omega_2 t} + C_2' e^{-i\omega_2 t}. \tag{4-157}$$

In order to make x_1 and x_2 real, we choose

$$C_1 = \tfrac{1}{2}A_1 e^{i\theta_1}, \qquad C_1' = \tfrac{1}{2}A_1 e^{-i\theta_1}, \tag{4-158}$$

$$C_2 = \tfrac{1}{2}A_2 e^{i\theta_2}, \qquad C_2' = \tfrac{1}{2}A_2 e^{-i\theta_2}, \tag{4-159}$$

$\delta = 0$

so that

$$x_1 = A_1 \cos(\omega_1 t + \theta_1) - \frac{\Delta\omega^2}{2\kappa^2}\sqrt{\frac{m_2}{m_1}}\, A_2 \cos(\omega_2 t + \theta_2), \quad (4\text{–}160)$$

$$x_2 = \frac{\Delta\omega^2}{2\kappa^2}\sqrt{\frac{m_1}{m_2}}\, A_1 \cos(\omega_1 t + \theta_1) + A_2 \cos(\omega_2 t + \theta_2). \quad (4\text{–}161)$$

This is the general solution, involving the four arbitrary constants A_1, A_2, θ_1, θ_2. We see that the motion of each coordinate is a superposition of two harmonic vibrations at frequencies ω_1 and ω_2. The oscillation frequencies are the same for both coordinates, but the relative amplitudes are different, and are given by Eqs. (4–154) and (4–155).

If A_1 or A_2 is zero, only one frequency of oscillation appears. The resulting motion is called a *normal mode of vibration.* The normal mode of
1 highest frequency is given by

$$x_1 = A_1 \cos(\omega_1 t + \theta_1), \quad (4\text{–}162)$$

$$x_2 = \frac{\Delta\omega^2}{2\kappa^2}\sqrt{\frac{m_1}{m_2}}\, A_1 \cos(\omega_1 t + \theta_1), \quad (4\text{–}163)$$

$$\omega_1^2 = \omega_{10}^2 + \tfrac{1}{2}\Delta\omega^2. \quad (4\text{–}164)$$

The frequency of oscillation is higher than ω_{10}. By referring to Fig. 4–10, we see that in this mode of oscillation the two masses m_1 and m_2 are oscillating out of phase; that is, their displacements are in opposite directions. The mode of oscillation of lower frequency is given by

2

$$x_1 = -\frac{\Delta\omega^2}{2\kappa^2}\sqrt{\frac{m_2}{m_1}}\, A_2 \cos(\omega_2 t + \theta_2), \quad (4\text{–}165)$$

$$x_2 = A_2 \cos(\omega_2 t + \theta_2), \quad (4\text{–}166)$$

$$\omega_2^2 = \omega_{20}^2 - \tfrac{1}{2}\Delta\omega^2. \quad (4\text{–}167)$$

In this mode, the two masses oscillate in phase at a frequency lower than ω_{20}. The most general motion of the system is given by Eqs. (4–160), (4–161), and is a superposition of the two normal modes of vibration.

The effect of coupling is thus to cause both masses to participate in the oscillation at each frequency, and to raise the highest frequency and lower the lowest frequency of oscillation. Even when both frequencies are initially equal, the coupling results in two frequencies of vibration, one higher and one lower than the frequency without coupling. When the coupling is very weak, i.e., when

$$\kappa^2 \ll \tfrac{1}{2}(\omega_{10}^2 - \omega_{20}^2), \quad (4\text{–}168)$$

then Eq. (4–150) becomes

$$\Delta\omega^2 \doteq \frac{2\kappa^4}{\omega_{10}^2 - \omega_{20}^2}. \tag{4–169}$$

For the highest frequency mode of vibration, the ratio of the amplitude of vibration of mass m_2 to that of mass m_1 is then

$$\frac{x_2}{x_1} = \frac{\Delta\omega^2}{2\kappa^2}\sqrt{\frac{m_1}{m_2}} \doteq \frac{\kappa^2}{\omega_{10}^2 - \omega_{20}^2}\sqrt{\frac{m_1}{m_2}}. \tag{4–170}$$

Thus, unless $m_2 \ll m_1$, the mass m_2 oscillates at much smaller amplitude than m_1. Similarly, it can be shown that for the low-frequency mode of vibration, m_1 oscillates at much smaller amplitude than m_2. If two oscillators of different frequency are weakly coupled together, there are two normal modes of vibration of the system. In one mode, the oscillator of higher frequency oscillates at a frequency slightly higher than without coupling, and the other oscillates weakly out of phase at the same frequency. In the other mode, the oscillator of lowest frequency oscillates at a frequency slightly lower than without coupling, and the other oscillates weakly and in phase at the same frequency. At or near resonance, when the two natural frequencies ω_{10} and ω_{20} are equal, the condition for weak coupling [Eq. (4–168)] is not satisfied even when the coupling constant is very small. $\Delta\omega^2$ is then given by Eq. (4–152), and we find for the two normal modes of vibration:

$\omega_{10} \neq \omega_{20}$

$\omega_{10} = \omega_{20}$

$$\frac{x_2}{x_1} = \pm\sqrt{\frac{m_1}{m_2}}, \tag{4–171}$$

$$\omega^2 = \omega_{10}^2 \pm \kappa^2. \tag{4–172}$$

The two oscillators oscillate in or out of phase with an amplitude ratio depending only on their mass ratio, and with a frequency higher or lower than the uncoupled frequency by an amount depending on the coupling constant.

An interesting special case is the case of two identical oscillators ($m_1 = m_2, k_1 = k_2$) coupled together. The general solution (4–160), (4–161) is, in this case,

$$x_1 = A_1 \cos(\omega_1 t + \theta_1) - A_2 \cos(\omega_2 t + \theta_2), \tag{4–173}$$

$$x_2 = A_1 \cos(\omega_1 t + \theta_1) + A_2 \cos(\omega_2 t + \theta_2), \tag{4–174}$$

where ω_1 and ω_2 are given by Eq. (4–172). If $A_2 = 0$, we have the high-frequency normal mode of vibration, and if $A_1 = 0$, we have the low-frequency normal mode. Let us suppose that initially m_2 is at rest in its equilibrium position, while m_1 is displaced a distance A from equilibrium

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and released at $t = 0$. The choice of constants which fits these initial conditions is

$$\theta_1 = \theta_2 = 0,$$
$$A_1 = -A_2 = \tfrac{1}{2}A, \tag{4–175}$$

so that Eqs. (4–173), (4–174) become

$$x_1 = \tfrac{1}{2}A\,(\cos \omega_1 t + \cos \omega_2 t), \tag{4–176}$$

$$x_2 = \tfrac{1}{2}A\,(\cos \omega_1 t - \cos \omega_2 t), \tag{4–177}$$

which can be rewritten in the form

$\sim \omega_{10}\, t$

$$x_1 = A \cos\left(\frac{\omega_1 - \omega_2}{2}\, t\right) \cos\left(\frac{\omega_1 + \omega_2}{2}\, t\right), \tag{4–178}$$

$$x_2 = -A \sin\left(\frac{\omega_1 - \omega_2}{2}\, t\right) \sin\left(\frac{\omega_1 + \omega_2}{2}\, t\right). \tag{4–179}$$

If the coupling is small, ω_1 and ω_2 are nearly equal, and x_1 and x_2 oscillate rapidly at the angular frequency $(\omega_1 + \omega_2)/2 \doteq \omega_1 \doteq \omega_2$, with an amplitude which varies sinusoidally at angular frequency $(\omega_1 - \omega_2)/2$. The motion of each oscillator is a superposition of its two normal-mode motions, which leads to beats, the beat frequency being the difference between the two normal-mode frequencies. This is illustrated in Fig. 4–11, where oscillograms of the motion of x_2 are shown: (a) when the high-frequency normal mode alone is excited, (b) when the low-frequency normal mode is excited, and (c) when oscillator m_1 alone is initially displaced. In Fig. 4–12, oscillograms of x_1 and x_2 as given by Eqs. (4–178), (4–179) are shown. It can be seen that the oscillators periodically exchange their energy, due to the coupling between them. Figure 4–13 shows the same motion when the springs k_1 and k_2 are not exactly equal. In this case oscillator m_1 does not give up all its energy to m_2 during the beats. Figure 4–14 shows that the effect of increasing the coupling is to increase the beat frequency $\omega_1 - \omega_2$ [Eq. (4–172)].

If a frictional force acts on each oscillator, the equations of motion (4–135) and (4–136) become

Damping

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + k_1'x_1 + k_3x_2 = 0, \tag{4–180}$$

$$m_2\ddot{x}_2 + b_2\dot{x}_2 + k_2'x_2 + k_3x_1 = 0, \tag{4–181}$$

(a)
(b)
(c)

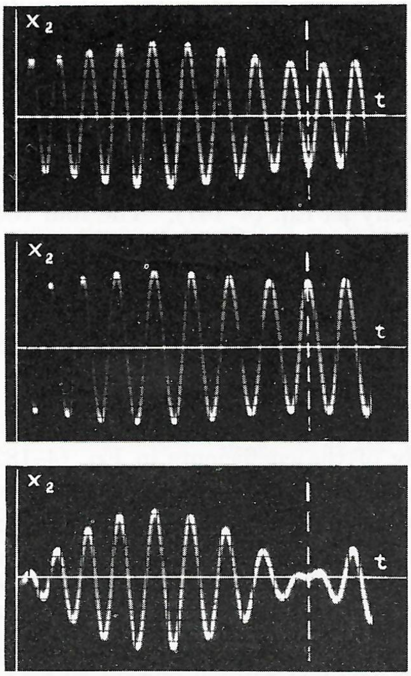

FIG. 4–11. Motion of coupled harmonic oscillators. (a) High-frequency normal mode. (b) Low-frequency normal mode. (c) m_1 initially displaced.

? Poincaré Recurrences ?

$\gamma = 0$

(a)
(b)

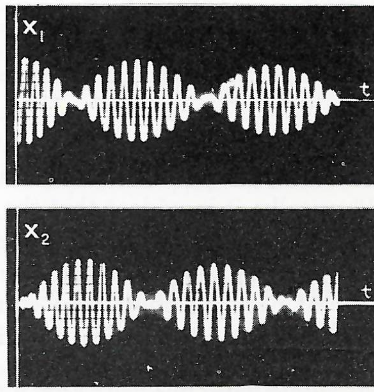

FIG. 4–12. Motion of two identical coupled oscillators.

$\gamma = 0$

(a)
(b)

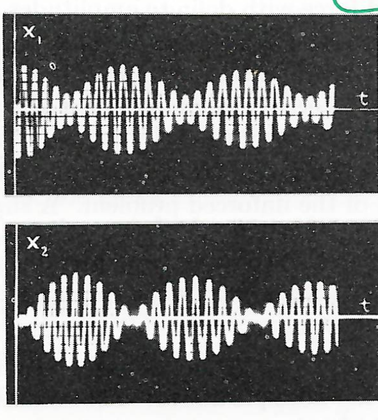

FIG. 4–13. Motion of two nonidentical coupled oscillators.

(a)
(b)

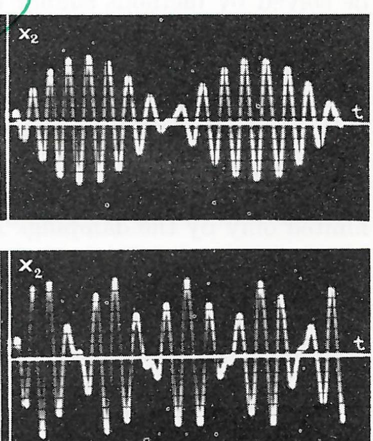

FIG. 4–14. Motion of two coupled oscillators. (a) Weak coupling. (b) Strong coupling.

δ

where b_1 and b_2 are the respective friction coefficients. The substitution (4–141), (4–142) leads to a fourth-degree secular equation for p:

$$m_1 m_2 p^4 + (m_2 b_1 + m_1 b_2)p^3 + (m_2 k'_1 + m_1 k'_2 + b_1 b_2)p^2 + (b_1 k'_2 + b_2 k'_1)p + (k'_1 k'_2 - k_3^2) = 0. \quad (4\text{–}182)$$

This equation cannot be solved so easily as Eq. (4–147). The four roots for p are, in general, complex, and have the form (if b_1 and b_2 are not too large)

$$p = -\gamma_1 \pm i\omega_1,$$
$$p = -\gamma_2 \pm i\omega_2. \quad (4\text{–}183)$$

That the roots have this form with γ_1 and γ_2 positive can be shown (though not easily) algebraically from a study of the coefficients in Eq. (4–182). Physically, it is evident that the roots have the form (4–183), since this will lead to damped vibrations, the expected result of friction. If b_1 and b_2 are large enough, one or both of the pairs of complex roots may become a pair of real negative roots, the corresponding normal mode or modes being overdamped. A practical solution of Eq. (4–182) can, in general, be obtained only by numerical methods when numerical values for the constants are given, although an approximate algebraic solution can be found when the damping is very small.

Driven

The problem of the motion of a system of two coupled harmonic oscillators subject to a harmonically oscillating force applied to either mass can be solved by methods similar to those which apply to a single harmonic oscillator. A steady-state solution can be found in which both oscillators oscillate at the frequency of the applied force with definite amplitudes and phases, depending on their masses, the spring constants, the damping, and the amplitude and phase of the applied force. The system is in resonance with the applied force when its frequency corresponds to either of the two normal modes of vibration, and the masses then vibrate at large amplitudes limited only by the damping. The general solution consists of the steady-state solution plus the general solution of the unforced problem. A superposition principle can be proved according to which, if a number of forces act on either or both masses, the solution is the sum of the solutions with each force acting separately. This theorem can be used to treat the problem of arbitrary forces acting on the two masses.

Other types of coupling between the oscillators are possible in addition to coupling by means of a spring as in the example above. The oscillators may be coupled by frictional forces. A simple example would be the case where one mass slides over the other, as in Fig. 4–15. We assume that the force of friction is proportional to the relative velocity of the two masses

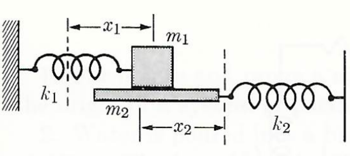

FIG. 4–15. Frictional coupling.

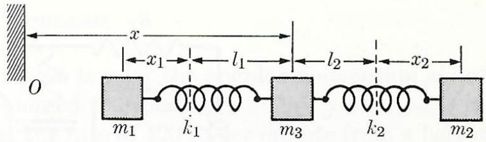

FIG. 4–16. Coupling through a mass.

The equations of motion of m_1 and m_2 are then

$$m_1\ddot{x}_1 = -k_1x_1 - b(\dot{x}_1 + \dot{x}_2), \tag{4–184}$$

$$m_2\ddot{x}_2 = -k_2x_2 - b(\dot{x}_2 + \dot{x}_1), \tag{4–185}$$

or

$$m_1\ddot{x}_1 + b\dot{x}_1 + k_1x_1 + b\dot{x}_2 = 0, \tag{4–186}$$

$$m_2\ddot{x}_2 + b\dot{x}_2 + k_2x_2 + b\dot{x}_1 = 0. \tag{4–187}$$

The coupling is expressed in Eqs. (4–186), (4–187) by a term in the equation of motion of each oscillator depending on the velocity of the other. The oscillators may also be coupled by a mass, as in Fig. 4–16. It is left to the reader to set up the equations of motion. (See Problem 26 at the end of this chapter.)

Two oscillators may be coupled in such a way that the force acting on one depends on the position, velocity, or acceleration of the other, or on any combination of these. In general, all three types of coupling occur to some extent; a spring, for example, has always some mass, and is subject to some internal friction. Thus the most general pair of equations for two coupled harmonic oscillators is of the form

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + k_1x_1 + m_c\ddot{x}_2 + b_c\dot{x}_2 + k_cx_2 = 0, \tag{4–188}$$

$$m_2\ddot{x}_2 + b_2\dot{x}_2 + k_2x_2 + m_c\ddot{x}_1 + b_c\dot{x}_1 + k_cx_1 = 0. \tag{4–189}$$

These equations can be solved by the method described above, with similar results. Two normal modes of vibration appear, if the frictional forces are not too great.

Equations of the form (4–188), (4–189), or the simpler special cases considered in the preceding discussions, arise not only in the theory of coupled mechanical oscillators, but also in the theory of coupled electrical circuits. Applying Kirchhoff's second law to the two meshes of the circuit shown in Fig. 4–17, with mesh currents i_1, i_2 around the two meshes as shown, we obtain

$$(L + L_1)\ddot{q}_1 + (R + R_1)\dot{q}_1 + \left(\frac{1}{C} + \frac{1}{C_1}\right)q_1 + L\ddot{q}_2 + R\dot{q}_2 + \frac{1}{C}q_2 = 0, \tag{4–190}$$

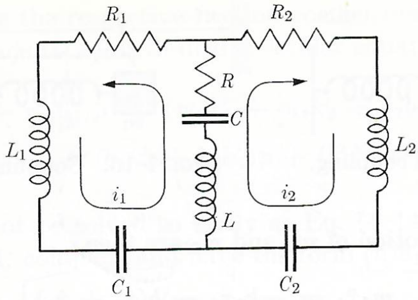

FIG. 4–17. Coupled oscillating circuits.

and

$$(L + L_2)\ddot{q}_2 + (R + R_2)\dot{q}_2 + \left(\frac{1}{C} + \frac{1}{C_2}\right) q_2 + L\ddot{q}_1 + R\dot{q}_1 + \frac{1}{C}\, q_1 = 0, \tag{4–191}$$

where q_1 and q_2 are the charges built up on C_1 and C_2 by the mesh currents i_1 and i_2. These equations have the same form as Eqs. (4–188), (4–189), and can be solved by similar methods. In electrical circuits, the damping is often fairly large, and finding the solution becomes a formidable task.

The discussion of this section can be extended to the case of any number of coupled mechanical or electrical harmonic oscillators, with analogous results. The algebraic details become almost prohibitive, however, unless we make use of more advanced mathematical techniques. We therefore postpone further discussion of this problem to Chapter 12.

All mechanical and electrical vibration problems reduce in the limiting case of small amplitudes of vibration to problems involving one or several coupled harmonic oscillators. Problems involving vibrations of strings, membranes, elastic solids, and electrical and acoustical vibrations in transmission lines, pipes, or cavities, can be reduced to problems of coupled oscillators, and exhibit similar normal modes of vibration. The treatment of the behavior of an atom or molecule according to quantum mechanics results in a mathematical problem identical with the problem of coupled harmonic oscillators, in which the energy levels play the role of oscillators, and external perturbing influences play the role of the coupling mechanism.