Electronic and Positronic Guiding-Center Drift Ions

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A novel type of guiding-center drift ion is described. These ions occur only in strong magnetic fields. They consist of a neutral atom to which either an electron or positron is weakly bound, at a sufficiently large radius that it may be described by $E \times B$ drift dynamics. Such ions may occur naturally in astrophysical plasmas and may have been formed in recent antihydrogen experiments, where their presence would provide proof that deeply bound $\bar{H}$ atoms are being created.

This Letter describes a novel type of ion that exists only in strong magnetic fields: a guiding-center ion. These ions consist of a neutral particle (an atom, molecule, or nanoparticle) to which a single electron or positron is bound at a distance large compared to the electron cyclotron radius. Stable guiding-center ions can be formed using any neutral particle, provided that the magnetic field is sufficiently large.

The outer electron or positron in a guiding-center ion executes $E \times B$ drift rotation around the neutral particle. The electric field arises from the attractive van der Waals interaction between the neutral particle and the outer charge, along with the dipole and quadrupole moments (if any) of the neutral particle. Without the magnetic field, these short-range interactions would lead to unstable classical orbits that either escape to infinity or spiral into the central neutral particle [1]. However, the applied magnetic field stabilizes the orbits.

It is conceivable that guiding-center ions may occur in strong magnetic fields associated with astrophysical phenomena such as neutron stars. A considerable effort over the years has gone into studying the state of neutral matter in such strong magnetic fields [2], and while there has been work on the properties of negative ions in such fields, little is known concerning positron attachment, as far as we know. The scaling of bound state negative ion energies has been considered [3], and there are several variational calculations of the ground state and the first few excited states of $\text{H}^-$ and other negative ions [4]. There has also been work describing positive ions (atoms that are missing electrons, not atoms with positrons attached) and ionic molecules [2,5].

Here, we focus on the regime of guiding-center ions where the dynamics of the outer charge is quasiclassical—i.e., the ion is excited well above the ground state. In this regime these ions have aspects in common with Rydberg atoms [6], a major difference being that here the interaction potential for the outer electron is not Coulombic.

The dynamics of these ions is similar to that of guiding-center hydrogen atoms, where an electron $E \times B$ drifts in the Coulomb potential of a central proton [7]. Antimatter guiding-center atoms have been observed to form in recent experiments that create atomic antihydrogen by recombination of positrons and antiprotons [8,9]. We will show that it is possible that guiding-center ions are also formed in these experiments, although they have not yet been observed.

We first examine a simple classical version of a guiding-center ion, consisting of a classical guiding-center atom to which an electron or positron of charge $e_2 = \pm e$ is attached at a cylindrical radius $r_2$. The electron in the atom, charge $e_1 = -e$, circles the central proton (assumed fixed at the origin) at a radius $r_1 < r_2$; but the presence of $e_2$ affects the $e_1$ orbit. A uniform magnetic field $B$ in the $z$ direction is assumed to be sufficiently strong so that the electron cyclotron frequency $\Omega_{ce}$ is larger than all other frequencies in the problem, allowing the guiding-center approximation to be applied. [Later we will relax this assumption, showing that it need not hold for the inner electron. We also assume that $r_1$ is much greater than the Bohr radius; conditions on $e_2$ for the validity of this classical description are discussed in relation to Eq. (3).]

The Hamiltonian for this system is

$$H(r_1, v_{z1}, r_2, v_{z2}) = \frac{1}{2m} v_{z1}^2 + \frac{1}{2m} v_{z2}^2 - \phi(|r_1|) \pm \phi(|r_2|) \pm \phi(|r_1 - r_2|),$$

(1)

where $r_1 = (r_1, \theta_1, z_1)$ and $r_2 = (r_2, \theta_2, z_2)$, $v_{z1}$ and $v_{z2}$ are the respective velocities parallel to the magnetic field, $\phi(r) = e^2/r$, and the upper and lower signs correspond to $e_2 = \pm e$. The equations of motion for $e_1$ and $e_2$ are $m \ddot{z}_i = -\partial H/\partial z_i$, $\dot{r}_i = -c/(e Br_i) \partial H/\partial \theta_i$, and $\dot{\theta}_i = c/(e Br_i) \partial H/\partial r_i$, $i = 1, 2$. This system has only two constants of the motion, the energy $H$ and the quantity $r_1^2 \pm r_2^2$, and so is not generally integrable. (However, if motion is restricted to the $xy$ plane, the dynamics is integrable, and will be considered in a future article.)

If $e_1$ is tightly bound at radius $r_1 \ll r_2$, the drift motion of $e_1$ is rapid, and we can make a two-time-scale approximation that considers the dynamics of $e_1$ assuming $e_2$ is stationary. This dynamics can then be used to find the motion of $e_2$, averaged over the rapid motion of $e_1$. For
example, one finds that to lowest nontrivial order, 
\( \theta_1 = -eEe/B \vec{r}_1 \equiv -\omega_1 \), and 
\( r_1 = \bar{r}_1 + r_2 \vec{r}_1 \cos(\theta_1 - \theta_2)/(r_2^2 + z_2^2)^{3/2} \), where \( \bar{r}_1 \) is the initial orbit radius of \( e_1 \). It is 
assumed here that \( \theta_1 = \theta_2 \) at \( t = 0 \). If we further assume 
that the \( z \) position of \( e_1 \) responds adiabatically to the 
to motion of \( e_2 \), neglecting the rapid axial bounce motion 
of \( e_1 \), the resulting slow-timescale motion of \( e_2 \) is 
Hamiltonian in form, with a Hamiltonian

\[
(H)(\vec{r}_2, v_{z2}) = \frac{1}{2} m v_{z2}^2 + eQ(z_2^2 - r_2^2)/2 
+ e^2 \alpha_z r_2^2 + \alpha_z z_2^2/(r_2^2 + z_2^2)^3/2 + O(r_1^4), \tag{2}
\]

where \( Q = e\bar{r}_1^2/2 \) is the axial quadrupole moment of a 
ring of charge \(-e\) with radius \( \bar{r}_1 \) (the time average of the 
\( e_1 \) orbit in the absence of \( e_2 \)) and \((\alpha_x, \alpha_z) = (5/4, 1/2)\bar{r}_1^2 \)
are coefficients in the van der Waals interaction between 
the guiding-center atom and \( e_2 \). This attractive interaction 
arises from the induced dipole moment of the atom in the 
electric field of \( e_2 \) [10].

If \( e_2 \) moves only in the \( xy \) plane, the charge’s binding 
energy (i.e., the value of \(-H\)) is 
\( E_b = -eQ/2r_2^3 + e^2\alpha_z r_2^3 \). The charge executes \( \vec{E} \times \vec{B} \)
drift rotation about the central atom with a rotation frequency 
\( \omega_2 = (eB)/(2Q/2r_2^3 + 4e\alpha_z e/r_2^3) \). If \( e_2 \) is a positron, 
the dynamics is stable in the \( z \) direction: a 
positron perturbed slightly from the \( xy \) plane will 
perform harmonic oscillations in \( z \) with frequency 
\( \omega_{z2} = [9eQ/2r_2^3 + (6\alpha_z - 2\alpha_z)e^2/r_2^3]/m \)^{1/2}. 

However, if \( e_2 \) is an electron, the quadrupole term is 
repulsive and \( z \) motion is stable at \( z = 0 \) only if 
\( r_2 \leq 4e(3\alpha_z - \alpha_z)/9Q = 26\bar{r}_1^3/9 \). For 
\( r_2 > 26\bar{r}_1^3/9 \), the outer electron is stable at an axial location 
determined by the potential minimum in \( H \) as a function 
of \( z_2 \) (the curve in Fig. 1). To the order given in \( \bar{r}_1 \), this 
minimum is at 
\( z_2 = (\sqrt{3}/2 r_2 - 19\bar{r}_1^3/3\sqrt{15}) \), 
with binding energy 
\( E_b = 2\sqrt{5}/e^2\bar{r}_1^2/25r_2^3 + 16e^2\bar{r}_1^4/125r_2^4 \), 
rotation frequency 
\( \omega_{z2} = (ecB)/(6\sqrt{5}/25r_2^4 + 64r_2^4/125r_2^4) \), 
and harmonic axial frequency 
\( \omega_{z2} = [(e^2/m)(36\sqrt{2}/5 \bar{r}_1^2/125r_2^3 + 1696\bar{r}_1^2/3125r_2^4)]^{1/2} \).

We have tested these predictions by numerically simulating 
a guiding-center ion using the guiding-center 
Hamiltonian of Eq. (1). Results are shown in Figs. 1 and 
2 for various values of the parameter 
\( \omega_1/\Omega_{ce} = mc^2/ \bar{r}_1^2 B^2 \). (This parameter must be less than unity in order for 
the guiding-center approximation to be valid for \( e_1 \).) By 
varying the initial \( r_2 \) value, we find stable guiding-center 
ions can exist with binding energy (Fig. 2), rotation 
frequency, and axial bounce frequency (not shown) that 
follow the previous predictions, provided that 
\( r_2/\bar{r}_1 \) is sufficiently large. (Binding energy is determined by 
slowly moving \( e_2 \) along a prescribed trajectory to its 
equilibrium position, starting from a large distance, 
and measuring the change in energy of the system.)

Furthermore, we find that, as expected, an outer positron 
is stable only in the \( xy \) plane, whereas an outer 
electron stably orbits the central particle at a height \( z_2 \) 
that follows the previous prediction (Fig. 1).

However, when \( r_2/\bar{r}_1 \approx 10 \), results begin to diverge 
noticeably from the theory predictions. There are initial 
values of \( r_2 > 26\bar{r}_1^3/9 \) where the outer electron is stable at 
\( z_2 = 0 \), at least over several hundred periods of the inner 
electron motion (Fig. 1). Furthermore, if \( r_2/\bar{r}_1 \) is too 
small initially, the guiding-center ion displays chaotic 
behavior. When \( e_2 \) is an electron, this typically results 
in its loss, with an increase in the binding of \( e_1 \) to 
compensate for the lost binding energy of \( e_2 \). When \( e_2 \) 
is a positron, the electron and positron often form a pair 
that \( \vec{E} \times \vec{B} \) drifts in parallel away from the central charge 
(a “drifting pair” [10]).

This chaotic behavior occurs because the two-timescale 
approximation breaks down: the most rapid dynamics of \( e_2 \) (i.e., the \( z \) motion) has roughly the same 
frequency as the rotational motion of the inner charge, 
i.e., \( \omega_{z2} \sim \omega_1 \). Using our previous theory expressions 
for these frequencies, and neglecting constants of 
order unity, this implies that the ion is stable only if 
\( r_2/\bar{r}_1 \approx (\Omega_{ce}/\omega_1)^{1/5} \).

FIG. 1 (color online). Axial equilibrium position for the outer 
electron in electronic classical guiding center ions. Curve is 
theory, dots are simulation results.

FIG. 2. Binding energy (affinity) for positron (upper curve) 
and electronic (lower curve) classical guiding center ions. Dots 
are simulation results, curves are theory.
The boundary between stable and unstable ions was found numerically by performing many simulations with different randomly chosen initial conditions and values of $\omega_1/\Omega_{ce}$, starting with both $z_1$ and $z_2$ within 0.01$F_1$ of the $xy$ plane and $|v_{1z}|$ and $|v_{2z}|$ less than 0.0005$eC/BF_1^2$. For any given initial value of $r_2/F_1$ and $\omega_1/\Omega_{ce}$, the fraction $f$ of times the outer charge was lost in a time 100/$\omega_1$ or less was computed over many simulations. For $e_2 = -e$, contours of constant $f$ are shown in Fig. 3. The $f = 0.8$ contour is fitted by $r_2/F_1 = 2.0(\Omega_{ce}/\omega_1)^{0.21}$, in close agreement to our previous estimate.

We now relax the assumption that the central positive charge is fixed at the origin. The behavior of the system now depends on the ratios $\Omega_{ci}/\omega_2$ and $V_i/(r_2 \omega_2)$, where $\Omega_{ci} = eB/M_c$ is the ion cyclotron frequency, and $M$ and $V_i$ are the mass and initial velocity of the central positive charge. When these ratios are small, the approximation that the central charge is stationary is a good one: $e_2$ executes roughly circular orbits about the central charge, and the entire system executes relatively slow circular ion cyclotron orbits with frequency $\Omega_{ci}$. However, if $V_i/(r_2 \omega_2) \gg 1$, the positive charge and $e_1$ run away from $e_2$, leaving it behind. In what follows we assume that $V_i = 0$; i.e., the central positive charge is initially stationary.

If $\omega_2/\Omega_{ci} \lesssim 1$, the ion cyclotron motion of the system is distorted, and in particular $r_2$ oscillates with time; the magnitude of these oscillations tends to increase with decreasing $\omega_2/\Omega_{ci}$. If during these oscillations $r_2/F_1$ decreases below the previously determined stability limit, $e_2$ is lost. Thus, stable guiding-center ions exist primarily in the range $\omega_2/\Omega_{ci} \gtrsim 1$. This rough inequality defines a maximum value of $r_2/F_1$ for stability: $r_2/F_1 \leq [C(M/m)\omega_1/\Omega_{ce}]^{1/5}$, where $C$ is a constant of order unity. Therefore, the heavier the central charge, or the larger the value of $\omega_1/\Omega_{ce}$, the larger the range of stable $r_2/F_1$ values.

This scaling was tested by running many simulations, allowing the central positive charge to move. Random initial conditions were integrated forward for times up to $10^5/\omega_1$. Initial values of $r_2$ and $\omega_1/\Omega_{ce}$ resulting in stable $H^-$ ions are shown on Fig. 3 as points. The clustering is consistent with our scaling results, taking $C = 5$ (the dashed line labeled $H^-$). Also shown are the scaling curves for $M = 4m_p$ and $131m_p$ (labeled $He^-$ and $Xe^-$).

One can see from Fig. 3 that stable $H^-$ ions exist primarily in the range $\omega_1/\Omega_{ce} \approx 0.2$, a range for which guiding-center dynamics is a poor approximation for $e_1$. However, even if $\omega_1/\Omega_{ce} > 1$, $e_2$ can still be described by guiding-center dynamics, and stable guiding-center ions can still be found. An example of a stable guiding-center $H^-$ ion for which $\omega_1/\Omega_{ce} = 10$ is shown in Fig. 4; now $e_1$’s dynamics is treated exactly, and motion of the central proton is allowed. For this simulation, the binding energy of the outer electron was determined to be 0.017 $e^2/F_1$, equal to 10 K when $B = 6T$.

Thus, the observation of $H^-$ ions in a recombining hydrogen plasma (or the antimatter equivalent) provides an indicator of the binding depth of the hydrogen (or antihydrogen) atoms: only if these atoms are bound such that $\omega_1/\Omega_{ce} \approx 0.2$ can $H^-$ ions form. In a 6 Tesla magnetic field, this inequality corresponds to an atomic binding energy $e^2/F_{mak\tilde{r}_1}$ deeper than 14 meV, roughly 5 times the binding energy estimated using $E$-field recombination diagnostics in current experiments [8,11].

For stability, the outer electron must be bound with energy $E_b \approx 0.04e^2/F_1 = 6 K$ when $e^2/F_1 = 14$ meV (see Fig. 2); and $E_b$ can be even greater for greater $e^2/F_1$. Thus, guiding-center ions may form in current antihydrogen experiments for which the plasma temperature $T_p$ satisfies $T_p \leq E_b$, provided that $e^2/F_1 \geq 14$ meV ($B/6T)^{2/3}$.

When $\omega_1/\Omega_{ce} \approx 1$, the inner electron dynamics is not integrable even in the absence of $e_2$. However, guiding-center ions may still form whenever $e_1$ is sufficiently deeply bound that all its dynamical frequencies are much higher than those of $e_2$. In this case the time-averaged Hamiltonian of Eq. (2) remains valid: the inner electron and the central positive charge act as a neutral...
particle with given quadrupole moment $Q$ and (positive) van der Waals coefficients $\alpha_x$ and $\alpha_z$. However, the values of these parameters must be determined numerically, and for very tight binding $e_1$ must be treated quantum mechanically; indeed, $e_2$ may require a quantum mechanical description as well if it is sufficiently tightly bound, even if its $xy$ motion is well described by guiding-center dynamics. We refer to such ions as quantum guiding-center ions.

We now consider the unusual wave function of quantum guiding-center ions. Assume that $Q = 0$ and $\alpha_x = \alpha_z = R^3$ (i.e., the central neutral particle is spherically symmetric, with radius of order $R$). Also, assume that $r_2 \gg \sqrt{\hbar/m\Omega_{cc}} \gg R$, where $\sqrt{\hbar/m\Omega_{cc}}$ is the quantum Larmor radius. Then the $xy$ motion of $e_2$ can be treated classically; i.e., the wave function for $e_2$ is localized at some given radius $r_2$. However, the axial motion of $e_2$ is strongly affected by quantum uncertainty.

Schrödinger’s equation for the $z$ motion of $e_2$ is

$$\frac{\hbar^2}{2m} \frac{\partial^2 \psi(z)}{\partial z^2} - \frac{e^2 R^3}{(r_2^2 + z^2)^2} \psi(z) = -E_b \psi(z), \quad (3)$$

where $\psi(z)$ is the axial wave function. Equation (3) can be put in dimensionless form by scaling distances by the length $a = \sqrt{R^3/a_B}$, and the energy by $\hbar^2/ma^2$, where $a_B = \hbar^2/me$ is the Bohr radius. Then Eq. (3) becomes

$$- \frac{1}{2} \frac{\partial^2 \hat{\psi}(\hat{z})}{\partial \hat{z}^2} - \frac{1}{(\hat{r}_2^2 + \hat{z}^2)^2} \hat{\psi}(\hat{z}) = -\hat{E}_b \hat{\psi}(\hat{z}), \quad (4)$$

where the hat denotes the use of scaled variables. The energy levels of the first three excited states are plotted in Fig. 5 versus $\hat{r}_2$, and the lowest wave function is plotted in the inset at $\hat{r}_2 = 10$. The energy levels can be understood analytically in two limiting cases: for $\hat{r}_2 \ll 1$, a Taylor expansion about $\hat{z} = 0$ gives a harmonic oscillator potential, with energy levels $\hat{E}_b = 1/\hat{r}_2^4 - \hat{\omega}_z^2 (n + 1/2)$, where $\hat{\omega}_z^2 = 2/\hat{r}_2^2$. This limit connects to the classical $z$ dynamics discussed previously [since $r_2/a = (r_2/R)\sqrt{a_B/R} \to 0$ in the classical limit].

However, for large $r_2$ ($r_2 > 0.8165$) only a single bound state exists. Asymptotic analysis of Eq. (4) implies that this energy level is $\hat{E}_b = \pi^2/8\hat{r}_2^6$ (the dashed line in Fig. 5), and the wave function is $\psi = e^{-\sqrt{2}\hat{E}_b |\hat{z}|}$ for $|\hat{z}| \gg 1$. This wave function is highly elongated along the $z$ direction due to quantum uncertainty, with an axial extent of order $a\hat{r}_2^3$. For example, consider an atom or molecule for which $R = 4a_B$ (a reasonable value for the ground state of many common atoms and molecules) and for which $r_2 = 10a = 80a_B$. The outer electron wave function (inset in Fig. 5) then extends to $|\hat{z}| \sim 1000$, i.e., $z \sim 8000a_B = 0.4 \mu m$.

Several aspects of these unusual ions remain to be addressed; such as the effect of cyclotron motion on ion stability, formation and reionization rates, and inclusion of radial dynamics in the quantum wave functions. These and other issues will be examined in forthcoming work.

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Note added in proof.—It has come to our attention that related work has considered the binding energy and stability of a central neutral particle in or near the ground state and a weakly bound electron [12]. This work also predicts the large $r_2$ limit of the binding energy curve shown in Fig. 5.