Geometric Electro-Magnetic Field Equations for Particles with Charge and Spin

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The complete and self-consistent "Maxwell" equations are displayed algebraically, describing (time-delayed) force and torque interactions between moving particles with Charge and Spin.

The Point

The algebra is the **4-level Grassmann linear algebra** describing 3D Euclidean space, with 4 Basis Elements **{ 3 Vectors, 1 Point }**, generating structures for bound particles, in addition to lines, planes, and volumes. Here, this is denoted as G3p1 .

With this full algebra, **Forces** from 2 or more particles can add to become a **Torque**, and two or more linear **Flows** can add to become a **Circulation** . This can not be expressed in the "tangent space" of Clifford Algebra Cl3, nor in standard Vector Algebra.

With similar clarity, the G3p1 algebra describes the motional "transformations" between vector electric $\vec{\mathrm{E}}$ and bi-vector magnetic $\widehat{\mathrm{B}}$**,** through 1st order ($\vec{\mathrm{v}}$ / c) motion of an Origin Point, *without* 2nd order space-time algebra effects.

Thus, the algebra explicitly distinguishes between conduction currents and the (orthogonal) spin/circulation currents; and between dynamic effects such as "spin-transfer" torque, and entropic effects such as conduction resistance or magnetization damping.

Duality : It Takes Two to Tango

The triumph of Maxwell's Equations is the description of Electro-Magnetic waves, propagating outward from particles at fundamental speed **c** . These are generally viewed as either "light waves" or particle-like "Photons", expressing the "wave / particle" duality of modern physics.

Geometrically, there are two fundamental lengths in electro-magnetism :
(1) the "classical radius of the electron" $\mathbf{R_e} = e^2/m_e c^2$ = 2.82 pm (10^{-15} m), which scales the electric interaction field E between *two* or more electric charges; and
(2) the "Compton wavelength", here denoted $\mathbf{D_v} = \hbar c/m_e c^2$ = 386. pm, which scales the magnetic dipole field B from a *single* electron Spin.

In G3p1, the EM wave emanating from a single particle is seen to be nilpotent (a "Photino"), with energy densiy $\mathrm{E}^2 + \mathrm{B}^2 = 0$. However, the *superposition of two* counter-propagating Photinos is seen to give the inter-particle Force, Torque and Helicity transfers generally attributed to a single Photon.

Understanding the complete geometry of points, vectors, bi-vectors, and tri-vectors can substantially clarify the meaning of motional transforms in space and time, and of complex probability waves.

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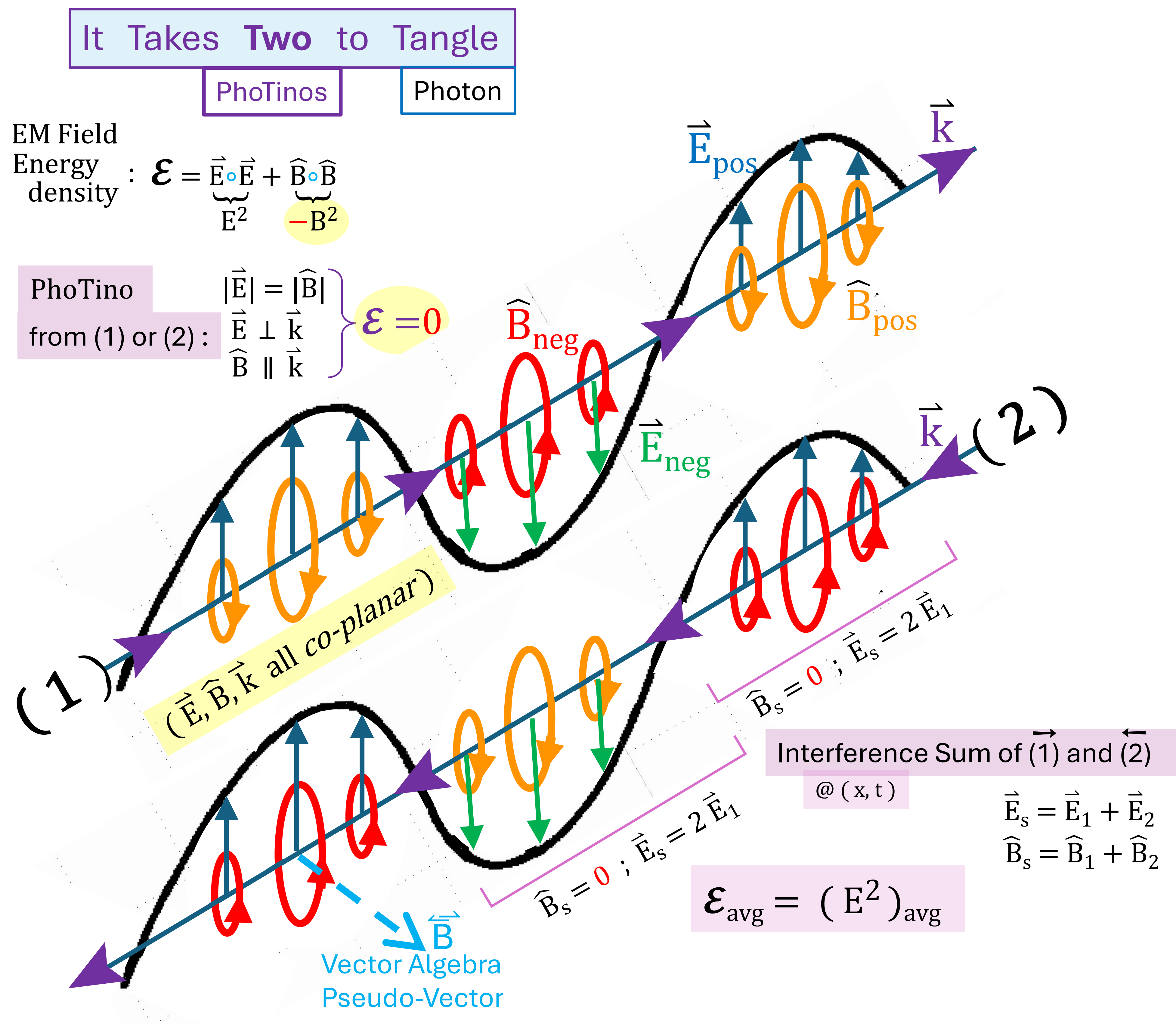
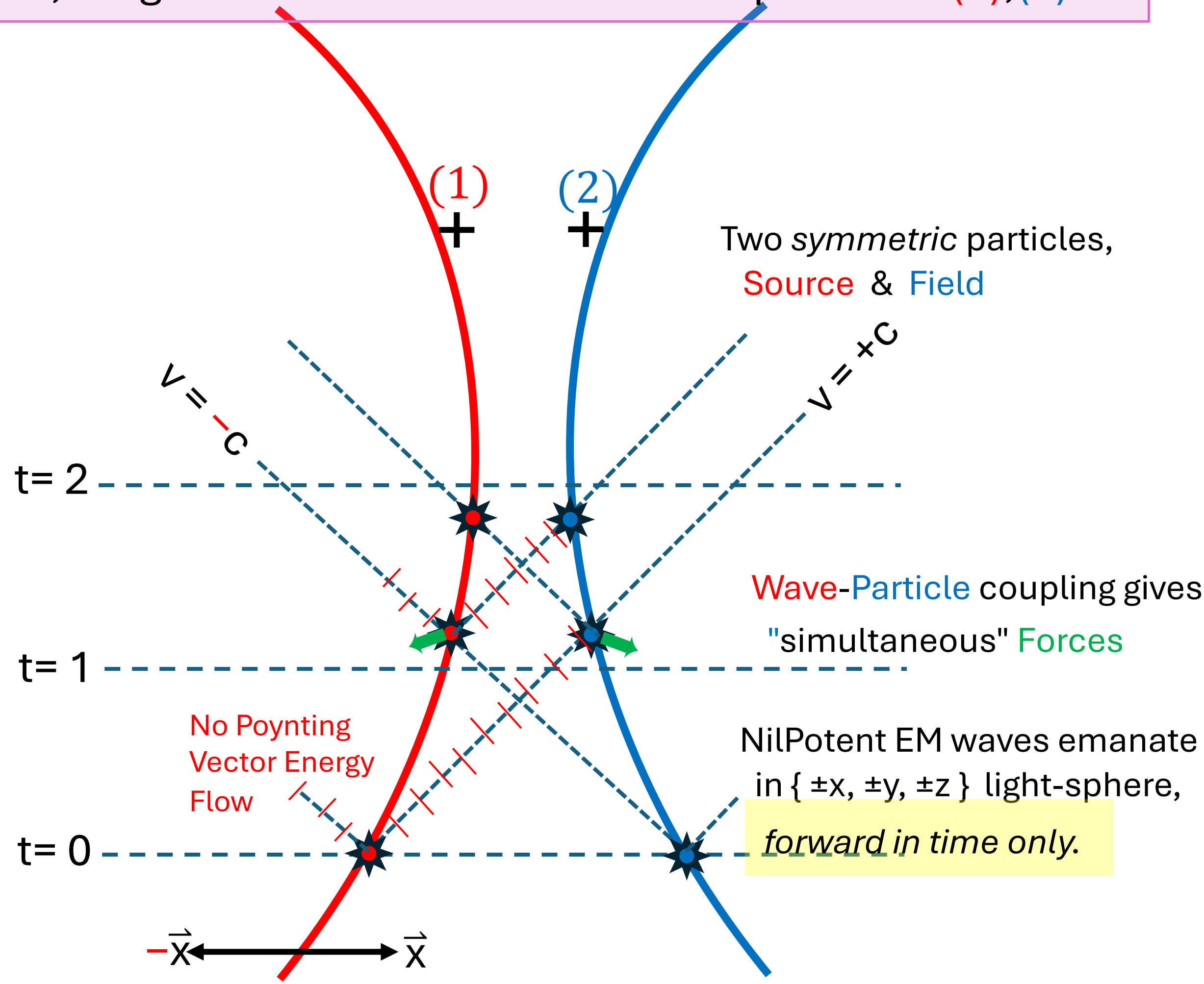

It Takes **Two** to Tangle
PhoTinos
Photon
EM Field Energy density : $\mathcal{E} = \underbrace{\vec{E}\circ\vec{E}}_{E^2} + \underbrace{\widehat{B}\circ\widehat{B}}_{-B^2}$
PhoTino from (1) or (2) :
$|\vec{E}| = |\widehat{B}|$
$\vec{E} \perp \vec{k}$
$\widehat{B} \parallel \vec{k}$
$\mathcal{E} = 0$
$\vec{E}_{pos}$
$\vec{k}$
$\widehat{B}_{pos}$
$\widehat{B}_{neg}$
$\vec{E}_{neg}$
(1)
(2)
$(\vec{E}, \widehat{B}, \vec{k}$ all co-planar)
$\widehat{B}_s = 0$; $\vec{E}_s = 2\,\vec{E}_1$
$\widehat{B}_s = 0$; $\vec{E}_s = 2\,\vec{E}_1$
Interference Sum of $\overrightarrow{(1)}$ and $\overleftarrow{(2)}$
@ (x, t)
$\vec{E}_s = \vec{E}_1 + \vec{E}_2$
$\widehat{B}_s = \widehat{B}_1 + \widehat{B}_2$
$\mathcal{E}_{avg} = (\,E^2\,)_{avg}$
$\overleftrightarrow{B}$
Vector Algebra
Pseudo-Vector

Particles (1),(2) emanate "NilPotent" spherical waves w1,w2 ;
waves w1,w2 give simultaneous Forces on particles (1),(2) .



Geometric Linear Algebras

Geometric Product : $\vec{A}\diamond\vec{B} = \vec{A}\circ\vec{B} + \vec{A}\wedge\vec{B}$
// ⊥

Gnp1 ≡ Grassmann Extension Algebra
n+1 Basis Elements : 1 **Point**, n **Vectors**
or : n+1 Points

n-Dimensional
Affine Space

𝒞ℓm ≡ Clifford "Geometric Algebra" (Hestenes)
m Basis Elements, all **Vectors**

Tangent Space

$\mathring{P}$ Point, Particle — $\vec{e}_1 = \mathring{P}_1 - \mathring{P}_0$ FreVec

spin $\mathring{P}\wedge\vec{e}$ BndVec — $\hat{U} = \vec{e}_1\wedge\vec{e}_2$ FreBiVec circulation

$\mathring{P}\wedge\hat{U}$ BndBiVec — $\tilde{T} = \vec{e}_1\wedge\vec{e}_2\wedge\vec{e}_3$ FreTriVec

$\mathring{P}\wedge\tilde{T}$ BndTriVec helicity

$\check{u} = \sqrt[2]{1}$

$\check{i} = \sqrt[4]{1}$

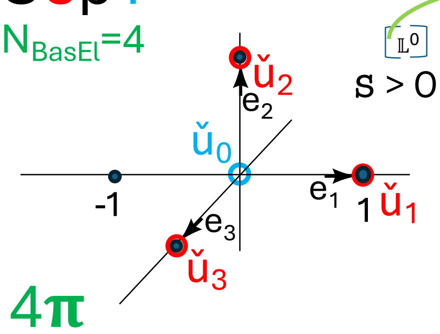
G3p1 gives full structures *needed* for geometric Particle/Field interactions in real 3-Space

Wedge, [Cross], Extension, Protrusion — ∧ ⊥

Dot, [Scalar] Contraction, Adherence — ∘ //

G3p1

$N_{BasEl}=4$



4π

3 SpcDim #16

g0 #1: $[\mathbb{L}^0]$; s > 0

g1 #4: ⊞ $[\mathbb{L}^{-3}, \mathbb{L}^1]$ ⊞
$\mathring{P}_0$
$\vec{e}_{01} = \mathring{P}_1 - \mathring{P}_0$ (02: 2, 0; 03: 3, 0)
carries: $\mathring{P}_0 + \vec{e}_{0i} = \mathring{P}_i$

g2 #6: ⊟ $[\mathbb{L}^{-2}, \mathbb{L}^2]$ ⊟
$\overleftarrow{R}_1 = \mathring{P}_0\wedge\vec{e}_1$ (2: 0, 2; 3: 0, 3)
$\hat{U}_{23} = \vec{e}_2\wedge\vec{e}_3$ (31: 3, 1; 12: 1, 2)
carries: $\overleftarrow{R}_{0i} + \hat{U}_{ji} = \overleftarrow{R}_{ji}$

g3 #4: ⊟ $[\mathbb{L}^{-1}, \mathbb{L}^3]$ ⊟
$\tilde{T}_{123} = \vec{e}_1\wedge\vec{e}_2\wedge\vec{e}_3$
$\hat{S}_{023} = \mathring{P}_0\wedge\hat{U}_{23}$ (031: 0, 31; 012: 0, 12)
carries: $\hat{S}_{0jk} + \tilde{T}_{ijk} = \hat{S}_{ijk}$

g4 #1: ⊞ $[\mathbb{L}^0]$
$\tilde{I}_4 = \mathring{P}_0\wedge\vec{e}_1\wedge\vec{e}_2\wedge\vec{e}_3$

$\check{u}_0\check{u}_1\check{u}_2\check{u}_3$

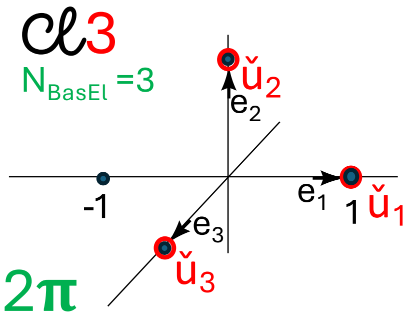
φ	$\vec{E}$	$\overleftarrow{E}_1 + \overleftarrow{E}_2 = \hat{B}_\perp + \overleftarrow{E}_3$	$\hat{M}$	$\tilde{H}$

$\overleftarrow{F}_1 + \overleftarrow{F}_2 = \hat{\tau}_\perp + \overleftarrow{F}_3$ Force, Torque

Tangent Space

𝒞ℓ3

$N_{BasEl}=3$



2π

#8

g0#1: ⊞ s ±

Pauli $S^{1/2}$

Quaternions = g0,g2

g1 #3: ⊞ $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$

g2 #3: ⊟ $\vec{e}_2\wedge\vec{e}_3$, $\vec{e}_3\wedge\vec{e}_1$, $\vec{e}_1\wedge\vec{e}_2$

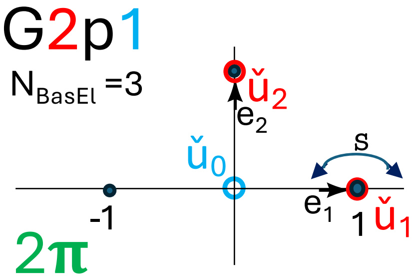
g3 #1: ⊟ $\tilde{I}_3 = \vec{e}_1\wedge\vec{e}_2\wedge\vec{e}_3$ ǐ

$\check{u}_1\check{u}_2\check{u}_3$

same

G2p1

$N_{BasEl}=3$



2π

2 SpcDim #8

g0 #1: s >0

g1 #3: ⊞ P_0; $P_1 - P_0 = \vec{e}_1$; $P_2 - P_0 = \vec{e}_2$; carries

g2 #3: ⊟ $\hat{U} = \vec{e}_1\wedge\vec{e}_2$; $\overleftarrow{R}_1 = P_0\wedge\vec{e}_1$ (2: 0, 2); carries

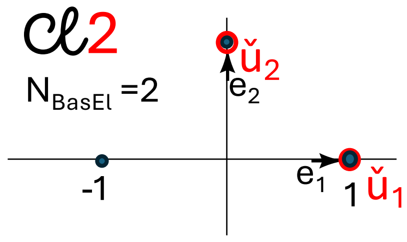
g3 #1: ⊟ $\hat{I}_3 = P_0\wedge\vec{e}_1\wedge\vec{e}_2$ ǐ

$\check{u}_0\check{u}_x\check{u}_y$

Tangent Space

𝒞ℓ2

$N_{BasEl}=2$



Complex: Z = g0,g2 ǐ

#4

g0 #1: s ±

g1 #2: $\vec{e}_1$ ⋮ $\vec{e}_2$

g2 #1: $\hat{I}_2 = \vec{e}_1\wedge\vec{e}_2$

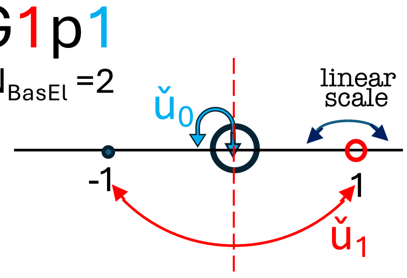
$\check{u}_x\check{u}_y$

Klein-4

same

G1p1

$N_{BasEl}=2$



1 SpcDim #4

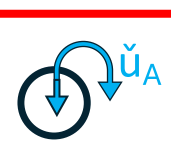
g0 #1: s >0

g1 #2: P_0 ⋮ $P_1 - P_0 = \vec{e}_1$; carries

g3 #1: $\hat{I}_2 = P_0\wedge\vec{e}_1$

$\check{u}_0\check{u}_x$

G0p1



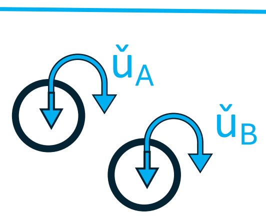
A = { 0 No / Out / Tails; 1 Yes / In / Heads } Dual

Not —

Boolean (Sets) #2

$\check{u} = \sqrt{1} = \{1, -1\}$

G0pm



(A, B independent)

#2^m

∧ or — $\overline{A \wedge B} = \overline{A} \circ \overline{B}$

∘ and — $\overline{A \circ B} = \overline{A} \wedge \overline{B}$

G3p1 linear algebra connects **Fields** to **Particle Scources** with **Charge** and **Spin**

$\mathbb{L}$, $\mathbb{T}$ $\rangle$ c defines Time

$\mathbb{M}$ $\quad \mathcal{E}_0 \equiv m_e c^2$

Length scales of E & M

$R_e = \dfrac{e^2}{\mathcal{E}_0} \quad \sim 2.82\ 10^{-15}\ \text{m}$

$D_v = \dfrac{\hbar c}{\mathcal{E}_0} \quad \sim 386.\ 10^{-15}\ \text{m}$

Cl3 Tangent | Fluid Maxwell | 2-fluids

g0 $\nabla \circ \vec{E} = \quad \nabla \circ \vec{E} = 4\pi(\rho_+ - \rho_-)$

g1 $-\nabla \circ \widehat{B} = \quad \nabla \times \overleftrightarrow{B} = \frac{\partial}{\partial ct}\vec{E} + 4\pi \vec{J}/c$

g2 $\nabla \wedge \vec{E} = \quad \nabla \times \vec{E} + \frac{\partial}{\partial ct}\overleftrightarrow{B} = 0$

g3 $\nabla \wedge \widehat{B} = \quad \nabla \circ \overleftrightarrow{B} = 0$

$e\boldsymbol{\varphi} \sim e^2 \mathbb{L}^{-1} \sim \mathcal{E}\text{nergy}$

$\vec{E} \sim e\mathbb{L}^{-2}$

$\widehat{B} \sim e\mathbb{L}^{-2} \sim e\mathbb{L}^{-1}\mathbb{T}^{-1}$

G3p1

$\check{u} = \{1, -1\} \qquad \check{u}_0\ \check{u}_x\ \check{u}_y\ \check{u}_z$

$q = \check{u}_Q\, e$

$\vec{r}_{sf} = \mathring{P}_s - \mathring{P}_f$

$\boxed{R = |\vec{r}_{sf}|}$

	e−	p+	e+	p−
$\check{u}_Q$	−	+	+	−
$\check{u}_H$	+	+	−	−

field @ (x_f, t) source @ (x_s, τ)

g0 [$\mathcal{E}$] $\quad e\boldsymbol{\varphi} = e\int dV \sum\left\{ \dfrac{q_s \mathring{P}_s}{R} + \dfrac{e\mathring{P}_s}{R}\dfrac{p_{s\parallel}}{R} \right\}_{\tau = t - R/c}$

g1 [$e\mathbb{L}^{-3}$] $\quad \nabla \circ \vec{E} = 4\pi \sum\left\{ q_s \mathring{P}_s + e\mathring{P}_s \dfrac{p_{s\parallel}}{R} \right\}_{\tau = t - R/c}$

g1 $\quad \nabla \circ \widehat{B} + \frac{\partial}{\partial ct}\vec{E} = 0$

g2 $\quad \nabla \wedge \vec{E} + \frac{\partial}{\partial ct}\widehat{B} = 0$

Waves : Oscillate Circulate

no $\vec{J}$

g3 $\quad \nabla \wedge \widehat{B} = 4\pi \sum\left\{ \check{u}_s \mathring{P}_s \wedge \dfrac{\vec{M}_{s\parallel}}{R} \right\}_{\tau = t - R/c}$

g4 [$\mathcal{E}$] $\quad e\widetilde{\mathcal{H}} = e\int dV \sum\left\{ \check{u}_s \mathring{P}_s \wedge \dfrac{\vec{M}_{s\parallel}}{R} \right\}_{\tau = t - R/c}$

1D-Dipole Charge

$\vec{p}_s = (\mathring{P}_{s+} - \mathring{P}_{s-}) \quad [\mathbb{L}^1]$

$p_{s\parallel} = \vec{p}_s \circ \vec{r}^{\,u}_{sf}$

2D-Dipole Magnetic Spin

$\widehat{M}_s = \frac{1}{2} D_v\, c\, \widehat{S}^u \quad [\mathbb{L}^2 \mathbb{T}^{-1}]$

$\vec{M}_{s\parallel} = \widehat{M}_s \circ \vec{r}^{\,u}_{sf}$

with $\widehat{S}^u_s = \check{u}_H (\vec{\beta}^u_s \circ \widetilde{T}^u)$

i.e. $\widehat{S}_s \perp \vec{\beta}_s$

Positional Energy   Spin 2D-Dipole Orientation   Charge 1D-Dipole Orientation

$\mathcal{E}/\mathcal{E}_0 = q_f \left(\boldsymbol{\varphi}(x_f) - \widehat{M}_f \circ \widehat{B} + \vec{d}_f \circ \vec{E} \right)$

Particle Force

$\vec{F}_f/\mathcal{E}_0 = q_f \left(\vec{E} - \widehat{M}_f \circ \nabla\widehat{B} + \vec{d}_f \circ \nabla\vec{E} \right)$

Particle Torque

$\hat{\boldsymbol{\tau}}_f/\mathcal{E}_0 = q_f \left(-\widehat{M}_f \wedge \widehat{B} + \vec{d}_f \wedge \vec{E} \right)$

EM Waves :

$\omega / \vec{k} = c$

$\tilde{f} = \sin(\vec{k} \circ \vec{r} - \omega t)[\vec{E} + \widehat{B}]$

$\vec{k} \perp \vec{E}, \quad \widehat{B} \parallel \vec{k}$

Invariants :

$\mathcal{E} = \vec{E}^2 + \widehat{B}^2$ Energy

$\widehat{\mathcal{H}} = \vec{E} \wedge \widehat{B}$ Helicity

Nilpotent "Photino"

$\widehat{B} = \check{u}_H \vec{k} \wedge \vec{E} \qquad \mathcal{E} = 0$

co-planar : $\widehat{B} \parallel \vec{E}$