Modulational instability of cnoidal wave solutions of the modified Korteweg–de Vries equation

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The stability of cnoidal wavetrain solutions of the modified Korteweg–de Vries equation is analyzed using Whitham's modulational theory. The cnoidal waves are solutions of an oscillator equation obtained by twice integrating the modified Korteweg–de Vries equation. The stability of the cnoidal waves is determined by the roots of the polynomial in the oscillator equation. For real roots the waves are stable, whereas for complex roots the waves are unstable.

I. INTRODUCTION

The Korteweg–de Vries (KdV) equation,[1]

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

characterizes the evolution of many systems with weak dispersion and quadratic nonlinearity.[2] Likewise, the modified Korteweg–de Vries (mKdV) equation,[3,4]

$$v_t \pm 12v^2 v_x + v_{xxx} = 0 \tag{2}$$

characterizes the evolution of systems with weak dispersion and cubic nonlinearity. For example, long wavelength disturbances on a one-dimensional lattice are described by the KdV equation when the restoring forces have a small quadratic nonlinearity, and by the mKdV equation when the restoring forces have a small cubic nonlinearity.[4]

The numerical coefficients in Eqs. (1) and (2) are arbitrary, since they may be changed by a change of scale (i.e., $x \to \alpha x$, $t \to \beta t$, $u \to \gamma u$, or $v \to \gamma v$). The coefficients 6 and 12 in front of the nonlinear terms in the two equations will be convenient for our purposes. The sign in front of the nonlinear term in Eq. (1) is arbitrary since it may be changed by the transformation $u \to (-u)$. The sign in front of the nonlinear term in Eq. (2) may not be changed by a real transformation, so we include the + or − possibilities explicitly. For the case of a nonlinear lattice this + or − sign corresponds to the sign of the cubic term in the restoring force.

Exact wavetrain solutions may be obtained for both equations.[1,5] By setting $u = u(x - Ct)$ in Eq. (1) and integrating twice with respect to x, one obtains

$$\tfrac{1}{2}u_x^2 + u^3 - \tfrac{1}{2}Cu^2 - Bu + A = 0, \tag{3}$$

where A and B are constants of integration. By following the same procedure with Eq. (2), one obtains

$$\tfrac{1}{2}v_x^2 \pm v^4 - \tfrac{1}{2}Cv^2 - Bv + A = 0, \tag{4}$$

where the constants A, B, and C do not necessarily have the same values in the two equations. These equations may be viewed as oscillator equations. The variable u oscillates back and forth between two roots of the polynomial in Eq. (3), and v oscillates between two roots of the polynomial in Eq. (4). Of course, these two roots must be real and adjacent, that is, not separated by another real root. Since the polynomials are cubic and quartic respectively, the equations can be integrated in terms of elliptic functions. The solutions are often called cnoidal waves, since they can be expressed in terms of the Jacobian elliptic function cn. When the modulus of the elliptic function is much smaller than unity the cnoidal waves reduce to sinusoidal waves, and when the modulus is near unity the cnoidal waves reduce to sequences of solitons.

The question of the stability of cnoidal waves was considered by Whitham for the case of the KdV equation.[6,7] He showed that these waves are stable to long wavelength perturbations, by applying his modulational theory.

Here, we apply Whitham's modulational theory to the case of the mKdV equation. We find that the question of the stability of a particular cnoidal wave depends on the values of the constants A, B, and C for that wave. The wave is stable if the polynomial in the associated oscillator equation [i.e., Eq. (4)] has four real roots and unstable if the polynomial has two real roots and two complex roots. A cnoidal wave can exist only if at least two roots are real, since in a cnoidal wave v oscillates back and forth between two real roots. From this perspective, one can understand Whitham's conclusion of stability for cnoidal wave solutions of the KdV equation. The polynomial in Eq. (3) is a real cubic, and the existence of two real roots implies that all three roots are real. The stability criterion may be stated in its most general form for the case of the generalized Korteweg–de Vries (gKdV) equation,

$$w_t + (6w \pm 12\mu^2 w^2)w_x + w_{xxx} = 0, \tag{5}$$

where μ is an arbitrary real constant determining the relative amount of quadratic and cubic nonlinearity. A cnoidal wave solution of this equation is stable if the roots of the polynomial in the associated oscillator equation,

$$\tfrac{1}{2}w_x^2 \pm \mu^2 w^4 + w^3 - \tfrac{1}{2}Cw^2 - Bw + A = 0, \tag{6}$$

are all real, and unstable if two roots are real and two are complex.

In Sec. II, we develop Whitham's modulational theory for the case of the mKdV equation. To be specific, we develop partial differential equations governing the temporal evolution of slow spatial modulations of the three parameters determining a cnoidal wave. In Sec. III, we find the Riemann invariants for the modulational equations. When the characteristic speeds for all three Riemann invariants are real the cnoidal wave is stable, and when the characteristic speeds are complex the cnoidal wave is unstable. The characteristic speeds are expressed in terms of the roots of the polynomial in the oscillator equation, and the real or complex nature of the characteristic speeds follows from that of the roots. In Sec. IV, we extend our results to the gKdV equation. In Sec. V, we discuss the relation of our results to the Miura transformation[3] between the KdV equation and the mKdV equation. The interpretation of this transformation will be seen to depend on the choice of the sign in the mKdV equation.

It is rather surprising that one can find the Riemann invariants for the modulational equations, that is, for three nonlinear coupled partial differential equations. Apparently, this is another example of the surprising degree to which problems associated with the KdV (or mKdV) equation yield to analytic methods.

II. MODULATIONAL EQUATIONS

Following Whitham[6,7] we derive the modulational equations by averaging conservation equations over a spatial oscillation of the cnoidal wave. The first three conservation equations for the mKdV equation are[8]

$$\begin{aligned}
&\frac{\partial}{\partial t}(v)+\frac{\partial}{\partial x}(\pm 4v^3+v_{xx})=0,\\
&\frac{\partial}{\partial t}(v^2)+\frac{\partial}{\partial x}(\pm 6v^4+2vv_{xx}-v_x^2)=0,\\
&\frac{\partial}{\partial t}(v^4\mp\tfrac{1}{2}v_x^2)+\frac{\partial}{\partial x}(\pm 8v^6+4v^3v_{xx}-12v^2v_x^2\mp v_xv_{xxx}\pm\tfrac{1}{2}v_{xx}^2)=0,
\end{aligned}\tag{7}$$

where the sign choice corresponds to that of Eq. (2).

Calculation of the average of quantities appearing in these equations is facilitated by introduction of the function

$$\begin{aligned}
W(A,B,C)&\equiv-\oint v_x\,dv\\
&=-\sqrt{2}\oint(-A+Bv+\tfrac{1}{2}Cv^2\mp v^4)^{1/2}\,dv,
\end{aligned}\tag{8}$$

where we have used Eq. (4) to find v_x for the cnoidal wave. The integral is defined to be over one complete cycle of the cnoidal wave. Since in a complete cycle v passes back and forth between two roots of the polynomial, the integral may be interpreted as a loop around the branch cut between the two roots. In terms of $W(A,B,C)$ the wavelength may be expressed as

$$\frac{1}{k}\equiv\lambda=\oint\frac{dv}{v_x}=\frac{\partial W}{\partial A}\equiv W_A,\tag{9}$$

and the average of v, v^2, and v_x^2 may be expressed as

$$\bar{v}=-kW_B,\quad \overline{v^2}=-2kW_C,\quad \overline{v_x^2}=-kW.\tag{10}$$

With the aid of Eq. (4) the average of all quantities in Eqs. (7) may be expressed in terms of the simple averages in Eqs. (9) and (10). The result is

$$\begin{aligned}
&\frac{\partial}{\partial t}(kW_B)+\frac{\partial}{\partial x}(kCW_B-B)=0,\\
&\frac{\partial}{\partial t}(kW_C)+\frac{\partial}{\partial x}(kCW_C-A)=0,\\
&\frac{\partial}{\partial t}[k(AW_A+BW_B+CW_C-W)]\\
&\quad+\frac{\partial}{\partial x}[kC(AW_A+BW_B+CW_C-W)-\tfrac{1}{2}B^2-AC]=0.
\end{aligned}\tag{11}$$

Simple algebraic reduction brings these equations to the more symmetric form

$$\begin{aligned}
\frac{D}{Dt}W_A&=W_A\frac{\partial C}{\partial x},\\
\frac{D}{Dt}W_B&=W_A\frac{\partial B}{\partial x},\\
\frac{D}{Dt}W_C&=W_A\frac{\partial A}{\partial x},
\end{aligned}\tag{12}$$

where

$$\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+C\frac{\partial}{\partial x}$$

is the convective derivative. These equations are the modulational equations for the mKdV equation. They differ from the corresponding equations for the KdV equation only in that W is defined in terms of the polynomial in Eq. (4) rather than that in Eq. (3).

III. RIEMANN INVARIANTS AND STABILITY CRITERION

We shall find that the Riemann invariants of the modulational equations take a simple form when expressed in terms of the roots of the polynomial in Eq. (4). From the relation

$$\pm v^4-\tfrac{1}{2}Cv^2-Bv+A\equiv\pm(v-a)(v-b)(v-c)(v-d)\tag{13}$$

we find that

$$\begin{aligned}
0&=a+b+c+d,\\
\tfrac{1}{2}C&=\mp(ab+ac+ad+bc+bd+cd),\\
B&=\pm(abc+abd+acd+bcd),\\
A&=\pm abcd.
\end{aligned}\tag{14}$$

By replacing the variables (A,B,C) by the variables (a,b,c), with d given by the first of Eqs. (14), the modulational equations take the form

$$\begin{aligned}
&W_{A,a}\frac{Da}{Dt}+W_{A,b}\frac{Db}{Dt}+W_{A,c}\frac{Dc}{Dt}\\
&\quad=\mp 2W_A[(d-a)a_x+(d-b)b_x+(d-c)c_x],\\
&W_{B,a}\frac{Da}{Dt}+W_{B,b}\frac{Db}{Dt}+W_{B,c}\frac{Dc}{Dt}\\
&\quad=\pm W_A[(b+c)(d-a)a_x+(a+c)(d-b)b_x+(a+b)(d-c)c_x],\\
&W_{C,a}\frac{Da}{Dt}+W_{C,b}\frac{Db}{Dt}+W_{C,c}\frac{Dc}{Dt}\\
&\quad=\pm W_A[bc(d-a)a_x+ac(d-b)b_x+ab(d-c)c_x],
\end{aligned}\tag{15}$$

where

$$\begin{aligned}
\frac{\partial W_A}{\partial a}\equiv W_{A,a}&=\frac{i}{\sqrt{\pm 8}}\oint\frac{(a-d)\,dv}{[(v-a)^3(v-d)^3(v-b)(v-c)]^{1/2}},\\
W_{B,a}&=\frac{-i}{\sqrt{\pm 8}}\oint\frac{(a-d)v\,dv}{[(v-a)^3(v-d)^3(v-b)(v-c)]^{1/2}},\\
W_{C,a}&=\frac{-i}{\sqrt{\pm 8}}\oint\frac{(a-d)\frac{1}{2}v^2\,dv}{[(v-a)^3(v-d)^3(v-b)(v-c)]^{1/2}}.
\end{aligned}\tag{16}$$

The quantities $W_{A,b}$, $W_{B,b}$, etc. are given by interchanging a and b in $W_{A,a}$, $W_{B,a}$, etc.

For the KdV case, the Riemann invariants are the various sums of roots taken two at a time (i.e., $a+b$, $b+c$, $a+c$). We now show that these quantities are Riemann invariants for the mKdV case as well. To show that $b+c$ is a Riemann invariant, we multiply the first of Eqs. (15) by $-(da+bc)(b+c)$, the second by $2(bc-da)$, the third by $-4(b+c)$, and add the three. The result is

$$\frac{i}{\sqrt{\pm 2}}\oint\left[\frac{v-a}{(v-d)^3(v-b)(v-c)}\right]^{1/2} dv\,\frac{D}{Dt}(b+c)$$

$$\pm 2W_A(a-b)(a-c)\frac{\partial}{\partial x}(b+c)=0, \tag{17}$$

where we have simplified the rhs of the equations with the identities

$$2(da+bc)(b+c)+2(b+c)(bc-da)-4bc(b+c)=0,$$

$$(d-b)[2(da+bc)(b+c)+2(a+c)(bc-da)-4ac(b+c)]$$

$$=-2(a-b)(b-d)(a-c)(c-d), \tag{18}$$

and the lhs with the identities

$$\frac{-(da+bc)(b+c)-2(bc-da)v+2(b+c)v^2}{[(v-a)^3(v-d)^3(v-b)(v-c)]^{1/2}}$$

$$=2\frac{d}{dv}\left[\frac{(v-b)(v-c)}{(v-d)(v-a)}\right]^{1/2},$$

$$\frac{(b-d)[-(da+bc)(b+c)-2(bc-da)v+2(b+c)v^2]}{[(v-b)^3(v-d)^3(v-a)(v-c)]^{1/2}} \tag{19}$$

$$=-2(b-d)\frac{d}{dv}\left[\frac{(v-a)(v-c)}{(v-b)(v-d)}\right]^{1/2}$$

$$+2(b-d)(c-d)\left[\frac{(v-a)}{(v-d)^3(v-b)(v-c)}\right]^{1/2}.$$

Note that b and c may be interchanged in all of these identities. Finally, we may rewrite Eq. (17) in the standard form

$$\frac{\partial}{\partial t}(b+c)+P\frac{\partial}{\partial x}(b+c)=0, \tag{20}$$

where the characteristic speed P is given by

$$P=C\pm\frac{2W_A(a-b)(a-c)}{(i/\sqrt{\pm 2})\oint[(v-a)/(v-d)^3(v-b)(v-c)]^{1/2}\,dv}$$

$$=C\pm\frac{2W_A(a-b)(a-c)}{W_A+2(d-a)(\partial/\partial d)(W_A)}. \tag{21}$$

Here, the partial derivative $\partial/\partial d$ must be taken before the first of Eqs. (14) is used to express d in terms of the other roots. By cyclic permutation of (a,b,c), one obtains the other two equations

$$\frac{\partial}{\partial t}(a+c)+Q\frac{\partial}{\partial x}(a+c)=0,$$
$$Q=C\pm\frac{2W_A(b-c)(b-a)}{W_A+2(d-b)(\partial/\partial d)(W_A)}, \tag{22}$$

and

$$\frac{\partial}{\partial t}(b+a)+R\frac{\partial}{\partial x}(b+a)=0,$$
$$R=C\pm\frac{2W_A(c-a)(c-b)}{W_A+2(d-c)(\partial/\partial d)(W_A)}. \tag{23}$$

We note that W_A may be expressed in terms of the elliptic integral of the first kind.[9] For the case of four real roots $(a>b>c>d)$, W_A is given by

$$W_A=\sqrt{8}\,K(r)/[(a-c)(b-d)]^{1/2}, \tag{24}$$

where $r^2=(a-b)(c-d)/(a-c)(b-d)$ for the upper sign choice in Eq. (2) and $r^2=(b-c)(a-d)/(a-c)(b-d)$ for the lower sign choice. For the case of two real roots $(c>d)$ and two complex roots $(b=a^*)$, W_A is given by

$$W_A=\sqrt{8}\,K(s)/\sqrt{pq}, \tag{25}$$

where

$$p^2=[c-(a+b)/2]^2-(a-b)^2/4,$$
$$q^2=[d-(a+b)/2]^2-(a-b)^2/4,$$
$$s^2=\frac{(c-d)^2-(p-q)^2}{4pq}.$$

This case is obtained only for the upper sign in Eq. (2).

To obtain the stability predictions of these equations, we follow the evolution of a small initial modulation [e.g., $b+c=b_0+c_0+(\delta b+\delta c)\cos(\kappa x)$], linearizing in the amplitude of the modulation [i.e., neglecting terms of order $(\delta b)^2$ or $(\delta c)^2$]. For the case of four real roots, the equations predict stability. For any solution, the wavelength W_A is real. Consequently, P, Q, and R are each real, and the small amplitude modulations oscillate rather than grow [e.g., $b+c=b_0+c_0+(\delta b+\delta c)\times\cos\{\kappa(x-Pt)\}$]. For the case of two real and two complex roots, the equations predict instability. Let c and d $(c>d)$ be the real roots, and a and b $(b=a^*)$ the complex roots. W_A is still real, but P and Q are complex, with $P=Q^*$. For a modulation with complex characteristic speed, either the component proportional to $\exp(i\kappa x)$ or the component proportional to $\exp(-i\kappa x)$ grows [e.g., $b+c=b_0+c_0+\frac{1}{2}(\delta b+\delta c)\exp\{i\kappa(x-Pt)\}+\frac{1}{2}(\delta b+\delta c)\exp\{-i\kappa(x-Pt)\}$]. This exponential growth will continue until the perturbation significantly modifies the wave parameters (a,b,c) and thus modifies the speeds (P,Q,R).

One could have anticipated that R is real and $Q=P^*$ from general considerations. Since the roots must occur in complex conjugates and since c and d are initially real and unequal, c and d must remain real during the initial evolution. This requires that R be real, since $a+b=-(c+d)$. Also, the evolution must preserve the relations $b=a^*$ or, since c is real, the relation $b+c=(a+c)^*$. This requires that $Q=P^*$. Of course, we could turn the argument around and show that the relations $R=R^*$ and $Q=P^*$ imply that the evolution preserves the relations $a=b^*$, $c=c^*$, $d=d^*$, and $a+b+c+d=0$.

For the case of complex characteristic speed, the equations take a more familiar form when rewritten in terms of real variables. If we let $X\equiv\mathrm{Re}(b+c)$, $Y\equiv\mathrm{Im}(b+c)$, $D\equiv\mathrm{Re}(P)$, and $E\equiv\mathrm{Im}(P)$, the real and imaginary parts of Eq. (20) are

$$\left(\frac{\partial}{\partial t}+D\frac{\partial}{\partial x}\right)X-E\frac{\partial}{\partial x}Y=0,$$
$$\left(\frac{\partial}{\partial t}+D\frac{\partial}{\partial x}\right)Y+E\frac{\partial}{\partial x}X=0. \tag{26}$$

Equation (22) leads to the same result, since $Q=P^*$. Eqs. (26) can be rewritten as the elliptic equations

$$\left(\frac{\partial}{\partial t}+D\frac{\partial}{\partial x}\right)^2 X+E^2\frac{\partial^2}{\partial x^2}X=0,$$
$$\left(\frac{\partial}{\partial t}+D\frac{\partial}{\partial x}\right)^2 Y+E^2\frac{\partial^2}{\partial x^2}Y=0, \tag{27}$$

and it is well known that elliptic equations are unstable for Cauchy boundary conditions.

For the simple example of small amplitude cnoidal waves, P, Q, and R may be explicitly evaluated as expansions in $c-d$, as shown in the Appendix. In this small amplitude limit, stability predictions may also be obtained from mode coupling theory,[10] for comparison with the modulational results. In the Appendix, we demonstrate that the characteristic speeds P, Q, and R agree with the stability results of mode coupling theory, to first order in $c-d$.

Finally, we note that the modulational equations describe the evolution of long wavelength perturbations only. For the small amplitude example, we are able to obtain higher order dispersive corrections from mode coupling theory. It is seen in the Appendix that these corrections tend to stabilize shorter wavelength perturbations.

IV. GENERALIZED KORTEWEG-DE VRIES EQUATION

In this section, we extend the results of the previous section to the gKdV equation. The first step is to note that the mKdV equation [i.e., Eq. (2)] is transformed into the gKdV equation [i.e., Eq. (5)] by the transformation

$$v=\mu w+1/(4\mu), \quad x\to x-3t/(4\mu^2). \tag{28}$$

Consequently, to every cnoidal wave solution of the gKdV equation there corresponds a cnoidal wave solution of the mKdV equation, and the stability (or instability) of the former may be inferred from that of the latter. By applying the same transformation to the oscillator equations for the two waves [i.e., Eqs. (4) and (6)], one can see that the roots of the polynomials in the two oscillator equations are also related by the transformation. Since this is a real transformation, we conclude that the cnoidal wave solution of the gKdV equation is stable when all four roots of the polynomial in the associated oscillator equation are real, and unstable when two roots are real and two are complex. Of course, the characteristic speeds for modulations and growth rates for instabilities are easily inferred from the transformation.

V. RELATION TO THE MIURA TRANSFORMATION

Miura's transformation[3] relates solutions of the mKdV equation, or gKdV equation, and solutions of the KdV equation. By setting $u=\pm 2v^2+\sqrt{\mp 2}\,v_x$ one can see by direct substitution that

$$u_t+6uu_x+u_{xxx}=\left(\pm 4v+\sqrt{\mp 2}\,\frac{\partial}{\partial x}\right)(v_t\pm 12v^2v_x+v_{xxx}). \tag{29}$$

Consequently, to every solution of the mKdV equation there corresponds a (possibly complex) solution of the KdV equation. The inverse does not follow because of the operator $(\pm 4v+\sqrt{\mp 2}\,\partial/\partial x)$ on the rhs.

For the lower choice of sign, the Miura transformation is real, and the stability properties of real solutions of the two equations should correspond. Consider the mKdV equation with negative nonlinear term. One can see from the associated oscillator polynomial [i.e., Eq. (4)] that bounded, real solutions exist only if all four roots are real. Thus our stability analysis shows that all real solutions of the mKdV equation with negative nonlinear term are stable, and this corresponds to the known stability of real solutions of the KdV equation.

For the upper choice of sign, the transformation is complex. The mKdV equation with positive nonlinear term has real, unstable solutions, obtained from oscillator polynomials with two real and two complex roots. These unstable mKdV solutions transform into complex, unstable solutions of the KdV equation. Of course, Whitham's stability analysis for the KdV equation was restricted to real solutions (as is ours for the mKdV equation), so the two results need not agree under a complex transformation.

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APPENDIX

In this appendix, we consider the simple example of small amplitude cnoidal waves; for clarity, we consider only the mKdV Eq. (2) with positive sign choice. We first obtain the stability results of mode coupling theory,[10] valid to first order in the wave amplitude and second order in the perturbation wavenumber κ. We then evaluate the modulational speeds P, Q, and R to first order in the wave amplitude. The two theories are seen to agree for long wavelength perturbations. For shorter wavelength perturbations, mode coupling theory gives corrections which tend to stabilize the growth of these components.

The small amplitude cnoidal wave is approximated by a mean value β, a fundamental mode A_1, and a single harmonic A_2. The perturbations are seen as sideband modes, with wavenumbers differing from the main modes by κ,

$$\begin{aligned} v(x,t)=\;&\beta+A_1\exp(ikx)+A_2\exp(2ikx)+A_+\exp(i\kappa x)\\ &+A_{1-}\exp[i(k-\kappa)x]+A_{1+}\exp[i(k+\kappa)x]\\ &+A_{2-}\exp[i(2k-\kappa)x]+A_{2+}\exp[i(2k+\kappa)x]\\ &+\text{complex conjugate.}\end{aligned} \tag{A1}$$

The evolution of a modal amplitude is determined by the appropriate spatial Fourier component of the mKdV equation. For component k this gives

$$\frac{\partial A_1}{\partial t}+i\,\omega_1 A_1+12ik(|A_1|^2A_1+2\beta A_2A_1^*)=0, \tag{A2}$$

where the linear frequency is $\omega_1 = -k^3 + 12\beta^2 k$. Solving the analogous evolution equations for the driven modes near $2k$ gives $A_2 = 4\beta A_1^2/k^2$, $A_{2-} = 8\beta A_1 A_{1-}/k^2$, $A_{2+} = 8\beta A_1 A_{1+}/k^2$. The nonlinear frequency of mode k is then seen to be $\Omega_1 = \omega_1 + 12k|A_1|^2(1+8\beta^2/k^2)$. The coupled evolution equations for the remaining perturbations are

$$\frac{\partial A_+}{\partial t} + i\omega_+ A_+ + 12i\kappa(2\beta A_{1+}A_1^* + 2\beta A_1 A_{1-}^*) = 0,$$

$$\frac{\partial A_{1-}}{\partial t} + i\omega_{1-}A_{1-} + 12i(k-\kappa)(2\beta A_{2-}A_1^* + 2\beta A_2 A_{1+}^* + 2\beta A_1 A_+^* + A_1^2 A_{1+}^* + 2|A_1|^2 A_{1-}) = 0, \tag{A3}$$

$$\frac{\partial A_{1+}}{\partial t} + i\omega_{1+}A_{1+} + 12i(k+\kappa)(2\beta A_{2+}A_1^* + 2\beta A_2 A_{1-}^* + 2\beta A_1 A_+ + A_1^2 A_{1-}^* + 2|A_1|^2 A_{1+}) = 0.$$

We take $A_1 \propto \exp(-i\Omega_1 t)$, $A_+ \propto \exp(-i\nu t)$, $A_{1-} \propto \exp(-i\Omega_1 t + i\nu t)$, $A_{1+} \propto \exp(-i\Omega_1 t - i\nu t)$, and solve Eqs. (A3) for the three roots ν. Two roots are seen to be near $\nu \approx \omega'\kappa = (-3k^2 + 12\beta^2)\kappa$; this approximation can be used to solve for $A_+ = -8\beta(A_{1+}A_1^* + A_1 A_{1-}^*)/k^2$. The resulting second order secular equation is

$$(\nu - \omega'\kappa)^2 = 12k|A_1|^2(1 - 8\beta^2/k^2)\omega''\kappa^2 + (\tfrac{1}{2}\omega''\kappa^2)^2, \tag{A4}$$

where $\omega'' = -6k$. A similar procedure gives the third root

$$\nu = 12\beta^2\kappa. \tag{A5}$$

The perturbation grows exponentially when one of the roots is complex, i.e., when the rhs of Eq. (A4) is negative.

We now evaluate P, Q, and R to order $c-d$, where $c > v > d$. To this order, Eqs. (24) and (25) for W_A are equivalent; we use Eq. (24) for simplicity,

$$W_A = \sqrt{2}\pi[(a-c)(b-d)]^{-1/2}(1 + r^2/4 + 9r^4/64).$$

Expressing all quantities in terms of $(a, c, c-d)$ gives

$$\frac{\partial}{\partial d}\ln(W_A) = \frac{c}{(a-c)(a+3c)} - \frac{(c-d)(10a^2 + 42c^2 + 4ac)}{16(a-c)^2(a+3c)^2},$$

$$k^2 \equiv \left(\frac{2\pi}{W_A}\right)^2 = -2(a-c)(a+3c) - 2(c-d)(3c-a),$$

$$|A_1| = \tfrac{1}{4}(c-d),$$

$$\beta = c - \tfrac{1}{2}(c-d).$$

The characteristic speeds may then be expressed as

$$\begin{aligned} P &= 6a^2 + 12ac - 6c^2 + (c-d)(3c-9a) \\ &= -3k^2 + 12\beta^2 - 6\sqrt{2}|A_1|(8\beta^2 - k^2)^{1/2}, \\ Q &= 6a^2 + 12ac - 6c^2 + (c-d)(9c-3a) \\ &= -3k^2 + 12\beta^2 + 6\sqrt{2}|A_1|(8\beta^2 - k^2)^{1/2}, \\ R &= 12c^2 - 12c(c-d) \\ &= 12\beta^2. \end{aligned} \tag{A6}$$

The two speeds P and Q, when multiplied by κ, correspond to the two roots ν in Eq. (A4); similarly, R times κ corresponds to the third root in Eq. (A5). The term $(\frac{1}{2}\omega''\kappa^2)^2$ in Eq. (A4) is a dispersive correction not found in modulational theory, and it decreases the instability for perturbations with large κ. Indeed, the small amplitude wavetrain is stable with respect to perturbations satisfying $\kappa^2 \geq 8|A_1|^2(1 - 8\beta^2/k^2)$. Thus modulational theory, valid for small κ, agrees with mode coupling theory, valid for small amplitude, in their range of overlap. Furthermore, mode coupling theory indicates that shorter wavelength perturbation tend to be stabilized.

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