Experimental test of modulation theory and stochasticity of nonlinear oscillations

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The initial nonlinear evolution of two unstable waves in a cold beam-plasma system can be described by a single wave with a slow modulated amplitude and phase. The modulational calculation is tested quantitatively on the analogous wave-particle interaction in a traveling wave tube. The model is found to be valid for one autocorrelation time following the onset of nonlinearities. Stochastic behavior follows the breakdown of the modulational approximation, i.e., after one autocorrelation time, in agreement with reasonance overlap theory.

I. INTRODUCTION

In the small cold beam limit, the beam plasma instability[1-5] is ideal for studying the development of stochasticity in an initially coherent nonlinear system. The beam drives a finite spectrum of waves unstable, but after many e foldings the bandwidth becomes narrow such that the beam dynamics are dominated by a single wave, namely, the fastest growing wave. This wave saturates by trapping the beam particles while forming clumps of particles in phase space. As the clumps oscillate in the wave potential, the wave amplitude and phase oscillate to conserve momentum and energy. If the wave were to remain monochromatic, as it is constrained to do in the single wave model computations,[2-4] the trapping oscillations would be long-lived indicating nonlinear stability for the clump.[6] However, experiments[5] and multiwave computations[7] show that the wave spectrum eventually broadens while the dominant wave decays in amplitude and the clumps are dispersed in phase space.

The randomization of the bucnhed particles is caused by the emerging sidebands consistent with resonance overlap theory.[6,8] According to this theory, stochastic behavior develops in a nonlinear system containing many waves when their trapping widths $\delta v_n = (e\phi_n/m)^{1/2}$, where ϕ_n is the amplitude of the nth wave, exceed the difference in their phase velocities Δv, i.e., when $\delta v_0 + \delta v_1 > 2\Delta v$. In the beam-plasma instability, most unstable waves satisfy this condition in the nonlinear regime. However, not all such waves are equally effective in inducing stochasticity because the randomization time for a particular satellite wave is inversely proportional to its frequency separation $\Delta\omega$ from the main wave. Consequently, the most deleterious satellite wave in a continuous spectrum are those farthest away in frequency from the main wave but which still satisfy the overlap condition. Nearby satellite waves do not play a significant role in the detrapping process because their mixing time is relatively long.

This point is exemplified in a calculation[9-10] of the nonlinear evolution of two waves with closely spaced frequencies in which the two waves were viewed as a single wave with a slowly varying amplitude and phase. For times less than an autocorrelation time, the modulation is slight and an individual particle "sees" an essentially monochromatic wave. The modulated wave can be thought of as a sequence of almost monochromatic waves, each differing in amplitude and phase from its predecessor. In this picture, the nonlinear dynamics are identical to those of a single wave and the evolution of the modulated wave is calculated by averaging the single wave response over the initial amplitude and phase. Results of this calculation have been verified experimentally,[9] but only for small relative amplitudes of the satellite wave. In addition, the limit on the applicability of the modulation approximation has not been investigated experimentally.

The present paper describes a quantitative test of the modulational calculation for arbitrary amplitude ratio $\epsilon \equiv \phi_1/\phi_0 \leq 1$ using a traveling wave tube[11-13] in which several trapped particle oscillations are observed. The experiments are conducted with a traveling wave tube because it has practical advantages over beam-plasma systems; theoretically, the interaction in a traveling wave tube is analogous to the beam-plasma instability. We find that for sufficiently small frequency separation, two launched waves saturate due to beam trapping and then execute trapping oscillations in quantitative agreement with the modulational calculation. The calculation fails when the particles resolve the spectral spread, namely, one autocorrelation distance beyond the onset of nonlinearities. For $\epsilon \approx 1$, the coherent trapping oscillations are destroyed immediately following the breakdown of the modulational approximation indicating stochastic behavior. For $\epsilon \ll 1$, the trapping oscillations persist long after the breakdown of the modulational approximation in accordance with resonance overlap theory.

The outline of the paper is as follows: Section II describes a quantitative comparison between experiment and the modulational theory. Section III discusses the breakdown of the theory and the onset of stochastic behavior. Section IV is the summary.

II. COMPARISON OF EXPERIMENTAL RESULTS WITH MODULATIONAL THEORY

The multiwave behavior of the beam-plasma instability is investigated through experiments on the analogous interaction in a traveling wave tube.[11] In the small cold beam limit the equations governing the evolution of the beam-plasma instability[2,3] are mathematically identical to those describing the traveling

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wave tube.[12] The plasma acts essentially as a linear dielectric capable of supporting slow space charge waves. Therefore, the replacement of the background plasma with the slow wave structure of a traveling wave tube does not alter the basic features of the wave particle interaction. The traveling wave tube, however, has two practical advantages in that the slow wave structure remains linear and does not introduce noise. With this system, which is described elsewhere,[13] over five trapping oscillations have been observed following exponential growth and saturation of a single launched wave.

The spatial evolution of two launched waves is shown in Fig. 1. For reference, the dotted line shows three trapping oscillations of the main wave at $\omega_0/2\pi = 195$ MHz when it is launched alone. The wave power, normalized to the injected beam power, agrees quantitatively (± 1 dB) with the results of the single wave model. The solid line is the main wave when we introduce an upper satellite (dashed line) at $\omega_{+1}/2\pi = 198$ MHz with an initial amplitude ratio $\epsilon = 0.39$. The main wave is perturbed only slightly by the presence of the additional wave, in agreement with the prediction of the modulational calculation[10] that the main wave is relatively unaffected for $\epsilon \leq 0.4$. When the satellite is launched alone, it saturates at the same amplitude as the main wave (-4.5 dB) because the detuning $\Delta\omega/\omega$ is small. However, in Fig. 1 the satellite saturates at a much smaller amplitude (-16.5 dB) near the point where the main wave traps the beam electrons. This agrees with previous beam-plasma experiments[9,14] in which a launched wave suppresses neighboring thermal noise below its natural level. Notice that the nonlinear product $\omega_{-1}/2\pi = 192$ MHz grows dramatically, at a rate much faster than the linear growth rate, when the dynamics become nonlinear, namely, within one e folding before saturation. This nonlinear product wave saturates at the same amplitude as the launched satellite in accordance with the modulational calculation[10] for small ϵ. Beyond saturation all the waves execute trapped particle oscillations with roughly the same bounce length $\lambda_b \approx 66$ cm, but at different phases in remarkable agreement with Fig. 2 of Ref. 10.

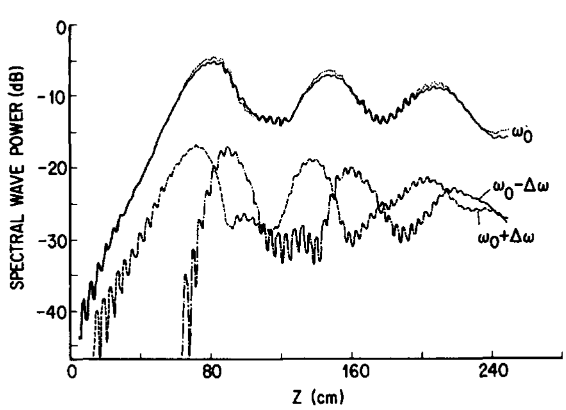

FIG. 1. Wave power normalized to injected beam power as a function of distance along the traveling wave tube. The dotted line is the main wave $\omega_0/2\pi = 195$ MHz when launched alone. The solid line is the main wave when the upper satellite (dashed line) is also launched with $\epsilon = 0.39$ and $\Delta\omega/2\pi = 3$ MHz. The intermodulational product (dot-dash) is also shown; the beam current is 10 mA; the beam voltage is 920 V.

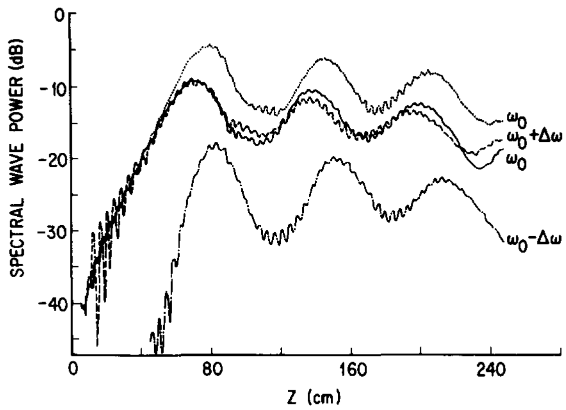

FIG. 2. Wave power normalized to injected beam power as a function of distance along the traveling wave tube. Conditions are same as in Fig. 1 except $\epsilon = 1.0$.

The fast oscillations in Fig. 1 are at half the wavelength. They result from the forward wave beating with a small component that is reflected by the imperfectly matched helix termination. The backward wave does not significantly affect the dynamics because it is not synchronous with the beam.

As predicted by modulational theory, the modes in Fig. 1 do not grow beyond the initial saturation points.[9,10] However, many intermodulational products with frequencies $\omega_n = \omega_0 \pm n\Delta\omega$, $n = 2, 3\ldots$ are nonlinearly unstable, but they remain relatively small within the length of the experiments and do not significantly affect the particle dynamics. In fact, there is little difference in the spatial evolution of the total wave power (measured with a braodband detector) between the single- and many-wave case because the total power in the satellite waves is small, roughly 10% of the total power.

The basic premise for the modulational calculation can be tested simply by launching the satellite wave with $\epsilon = 1.0$ as shown in Fig. 2. In particular, two waves launched with equal amplitude should evolve similarly as long as the beam particles do not resolve the frequency separation. Indeed, this occurs in Fig. 2 and since $\Delta\omega$ is small, the two launched waves evolve together over the entire interaction region. Their saturation amplitude is 4.5 dB lower than the saturation level for a single wave. The wave at ω_{-1} appears spontaneously near the onset of nonlinearities and saturates 9.5 dB below the saturation level of the main wave. Once again, many intermodulation products appear following saturation but 80% of the wave power remains in the two launched waves.

For quantitative comparison with the modulation calculation, we plot the saturation amplitudes of the main wave, launched satellite and the nonlinear product as a function of ϵ in Fig. 3. The amplitudes are normalized to the saturation level of the main wave when it is launched alone. The measured points agree with the calculation (solid line) within the experimental error (± 1 dB). For $\epsilon \leq 0.4$, the saturation amplitudes for the primary satellite waves are equal and proportional to ϵ

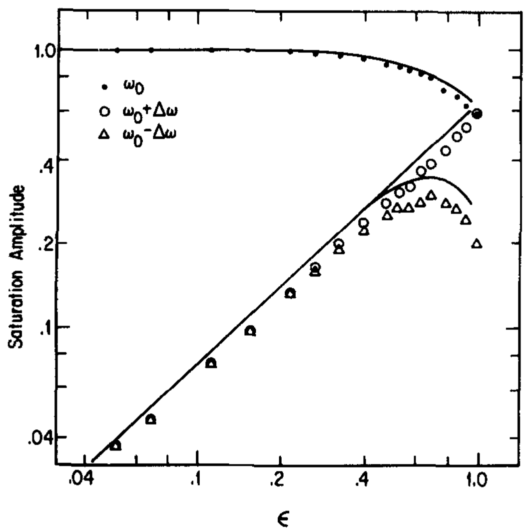

FIG. 3. Saturation amplitude for main wave $\omega_0/2\pi = 195$ MHz, launched satellite $\omega_0 + \Delta\omega(\Delta\omega/2\pi = 3$ MHz), and intermodulational product $\omega_0 - \Delta\omega$ as a function of initial amplitude ratio ϵ. Saturation amplitudes are normalized to the saturation amplitude of the main wave when it is launched alone. The beam current is 10 mA; the beam voltage is 920 V.

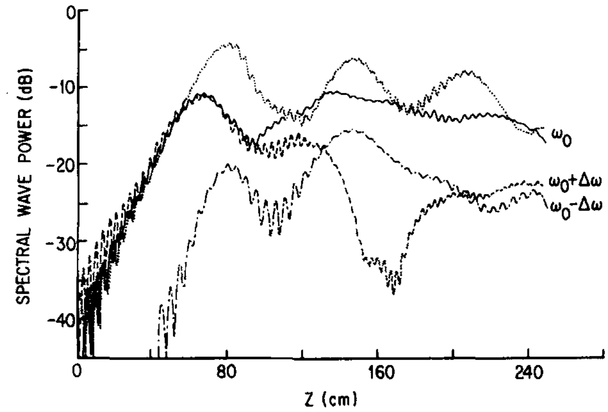

FIG. 4. Wave power versus distance along the traveling wave tube. Conditions same as in Fig. 2 except $\Delta\omega/2\pi = 25$ MHz.

while the evolution of the main wave is unaffected. For $\epsilon > 0.4$, the satellites play an active role in trapping the particles. Consequently, their saturation amplitudes are no longer simply proportional to ϵ, and the saturation level of the main wave decreases with increasing ϵ. Although the saturation amplitudes of the individual waves depend on ϵ, the total wave saturation power (broadband measurement) varies only slightly (± 0.5 dB) with ϵ.

Figure 3 shows that the modulational calculation correctly predicts the saturation amplitudes when two waves are launched. We emphasize that the calculation, which is described in detail in Ref. 10 and repeated here for different values of ϵ, is performed for a small cold beam-plasma system. However, it is directly applicable to the traveling wave tube because the interactions in both systems are analogous.

III. BREAKDOWN OF MODULATIONAL THEORY

The breakdown of the modulational approximation can be observed by increasing $\Delta\omega$. In Fig. 4, the conditions are identical to those in Fig. 2 except $\Delta\omega/2\pi = 25$ MHz. The waves at ω_0 and ω_{+1} are launched with equal amplitude, and they evolve together until $z = 90$ cm. At this point, the two launched waves depart dramatically signifying the breakdown of the modulational approximation.

The dependence of the breakdown position z_b on the frequency separation is shown in Fig. 5. A linear relation between z_b and $\omega/\Delta\omega$ is observed which can be expressed as

$$z_b \approx z_0 + 0.6 L_a , \qquad (1)$$

where L_a is the autocorrelation distance, which is the distance a resonant particle must travel in the laboratory to resolve the frequency separation $\Delta\omega$, and is given by

$$L_a \approx \frac{\pi v_0}{\Delta\omega(v_0/v_g - 1)} , \qquad (2)$$

where v_0 is the phase velocity of the main wave, and v_g is the group velocity ($v_0/v_g \approx 1.6$ for the experimental results shown here). The zero intercept in Fig. 5 $z_0 = 60$ cm is the point where the dynamics become nonlinear as indicated by the rapid emergence of the nonlinear product wave in Figs. 1, 2, and 4. Equation (1) shows that the modulation approximation fails when the resonant particles resolve the spectral spread following the onset of nonlinearities.

For two waves of comparable amplitude, the breakdown of the modulational approximation marks the transition from coherent to stochastic behavior as evidenced by the washing out of the trapping oscillations in Fig. 4. Such a stochastic instability arises when the trapping widths of two waves exceed the difference in phase velocities; specifically, this so-called resonance overlap condition is[8]

$$\delta v_0 + \delta v_1 > 2|v_0 - v_1| , \qquad (3)$$

where v_n is the phase velocity of the nth satellite wave. In the beam-plasma instability as well as in a traveling wave tube, most linearly unstable modes have a phase

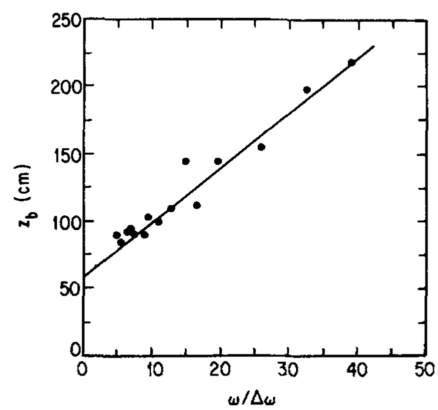

FIG. 5. Breakdown position of the modulational approximation as a function of $\omega_0/\Delta\omega$. Main wave and upper satellite are launched with $\epsilon = 1.0$. The beam current is 10 mA; the beam voltage is 920 V.

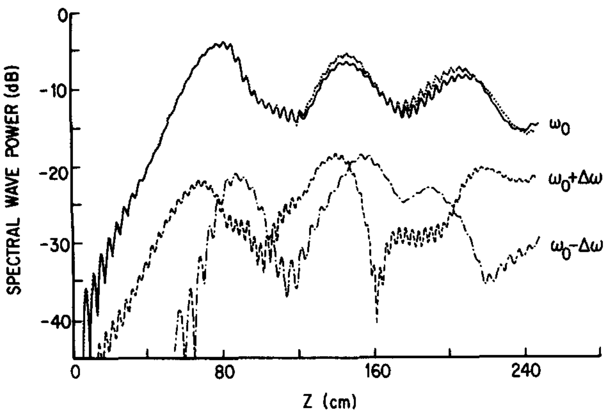

FIG. 6. Wave power versus distance along the traveling wave tube. The conditions are the same as in Fig. 4 except $\epsilon = 0.2$.

velocity within a trapping width of the saturated main wave and, therefore, they satisfy the overlap criterion in the nonlinear region.

When Eq. (3) is satisfied, the phase space trajectories of neighboring particles diverge in a diffusion time which, in the laboratory frame, transforms to a distance z_d given by

$$z_d - z_0 \approx L_a/\pi\epsilon_s^2 \ln K, \tag{4}$$

where $K \equiv [\delta v/v_0 - v_1)]^2$ is the K entropy.[8] ϵ_s is the saturated amplitude ratio ϕ_1/ϕ_0 and is related to ϵ according to Fig. 3. z_0 is included in Eq. (4) because particle diffusion is significant only after the dynamics become nonlinear.

By comparing Eqs. (1) and (4), we find that stochastic behavior occurs immediately following the breakdown of the modulational calculation when the two waves have comparable amplitudes $\epsilon_s \sim 1$. However, for $\epsilon_s \ll 1$, the coherent motion of the bunched particles persist beyond an autocorrelation distance and this is shown in Fig. 6. The conditions in Fig. 6. are identical to those in Fig. 4 except that the satellite is launched with a smaller amplitude so that $\epsilon_s \sim 0.13$. The trapping oscillations of the main wave persist throughout the length of the experiment, that is, far beyond the autocorrelation distance $L_a \sim 30$ cm even though the overlap criterion Eq. (3) is still satisfied. This is consistent with theory because Eq. (4) predicts that the mixing length is $z_d - z_0 \sim 17$ m due to the smallness of ϵ_s. Conversely, the position where the modulational approximation fails is relatively unchanged by the decrease in ϵ_s as indicated by the growth of the satellite waves during the second trapping oscillation in Fig. 6. Remember, according to the modulational calculation, the satellite waves should be nonlinearly stable (see Fig. 1) and the nonlinear growth of these waves indicates the breakdown of the modulational calculation.

IV. SUMMARY

In summary, we have found that a modulational calculation successfully describes the initial nonlinear evolution of two unstable waves in a system analogous to the beam-plasma system for all relative amplitudes of the waves. The model fails after an autocorrelation distance beyond the onset of nonlinearities, that is, after the particles resolve the frequency separation of the two waves. Since most satellite waves within the instability bandwidth satisfy the resonance overlap condition, they induce stochasticity and detrap the particles. The mixing time is found to be proportional to the autocorrelation time. Since the autocorrelation time is inversely proportional to $\Delta\omega$, satellite waves close in frequency to the main wave are not very effective in inducing stochasticity even though they satisfy the overlap condition. This also suggests that the most deleterious satellite waves in a continuous spectrum are those farthest away in frequency from the main wave but which still overlap with the main wave.

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