

# Collisional dynamics of a strongly magnetized pure electron plasma

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For a pure electron plasma in a sufficiently strong magnetic field, there is a many-electron adiabatic invariant which constrains the collisional dynamics. For the case of a uniform magnetic field, the adiabatic invariant is the total kinetic energy associated with the electron velocity components that are perpendicular to the magnetic field (i.e.,  $\sum_j mv_{j\perp}^2/2$ ). Were the adiabatic invariant an exact constant of the motion, no exchange of energy would be possible between the parallel and the perpendicular degrees of freedom, and the plasma could acquire and maintain two different temperatures,  $T_{\parallel}$  and  $T_{\perp}$ . However, an adiabatic invariant is not strictly conserved. In the present case, each collision produces an exponentially small exchange of energy between the parallel and the perpendicular degrees of freedom, and these act cumulatively in such a way that  $T_{\parallel}$  and  $T_{\perp}$  relax to a common value. This paper provides a calculation of the equipartition rate.

## I. INTRODUCTION

A recent paper<sup>1</sup> established the existence of a many-electron adiabatic invariant that constrains the collisional dynamics of a strongly magnetized pure electron plasma. For the case of a uniform magnetic field, the invariant is simply the total kinetic energy associated with the electron velocity components perpendicular to the magnetic field (i.e.,  $\sum_j mv_{j\perp}^2/2$ ). This paper discusses the influence of the invariant on the long-time collisional evolution of the electron velocity distribution.

Because of the invariant, the evolution proceeds on two very different time scales. The usual time scale for such evolution is the time for a few collisions. On this time scale, the adiabatic invariant is well conserved, and there is negligible exchange of energy between the parallel and the perpendicular degrees of freedom. Thus the distribution of parallel velocities and the distribution of perpendicular velocities become Maxwellian separately, with the parallel temperature ( $T_{\parallel}$ ) not necessarily equal to the perpendicular temperature ( $T_{\perp}$ ).

The evolution does not stop at this stage, since an adiabatic invariant is not strictly conserved; it suffers exponentially small changes. In the present case, each collision produces an exponentially small exchange of energy between the parallel and the perpendicular degrees of freedom, and these act cumulatively in such a way that  $T_{\parallel}$  and  $T_{\perp}$  relax to a common value. This paper provides a calculation of the rate at which this exponentially slow equipartition proceeds.

In Sec. II, the rate is calculated for the case where the electron dynamics may be described by classical mechanics. The analysis begins with the consideration of an isolated collision between two electrons that interact electrostatically in the presence of a strong and uniform magnetic field  $\mathbf{B}$ . The magnetic field is strong in the sense that  $\Omega \gg |v_j|/b$ , where  $\Omega = eB/mc$  is the cyclotron frequency,  $v_j$  is the initial velocity of electron  $j$  ( $j = 1, 2$ ), and  $b = e^2/(\mu v_{\parallel}^2/2)$  is the distance of closest approach between the two electrons. Here,  $\mu = m/2$  is the reduced mass, and  $v_{\parallel}$  is the initial parallel relative velocity. The orbits for such a collision are quite different

from those for Rutherford scattering. The Larmor radii for the two electrons are small compared to the distance between the electrons, and the electrons spiral toward and away from one another along tight helical orbits that follow field lines.

We find that the exchange of parallel and perpendicular energy that occurs during such a collision is of the order  $\exp(-\Omega\tau)$ , where  $\tau$  is a time that characterizes the duration of the collision. This time is a function of the initial parallel relative velocity  $v_{\parallel}$  and of the distance  $\rho$  between the field lines on which the electrons move; one can think of this distance as a kind of impact parameter. The time  $\tau$  is shortest, and the energy exchange is largest, for collisions characterized by small impact parameter (i.e.,  $\rho \ll b$ ). For such collisions, the time is given by  $\tau = (\pi/2)(b/v_{\parallel})$ , and the energy exchange is of the order  $\exp[-(\pi/2)(\Omega b/v_{\parallel})]$ . This quantity is exponentially small, since the condition for strong magnetization ensures that the parameter  $\kappa = \Omega b/v_{\parallel}$  is large compared to unity.

For simplicity, we consider the case of a weakly correlated plasma, and we say that such a plasma is strongly magnetized when  $\Omega \gg (T_{\max}/m)^{1/2}/\bar{b}$ , where  $T_{\max}$  is the larger of  $T_{\parallel}$  and  $T_{\perp}$  and  $\bar{b}$  is the quantity  $b$  with  $v_{\parallel}$  replaced by  $\bar{v}_{\parallel} = \sqrt{T_{\parallel}/\mu}$  (i.e.,  $\bar{b} = 2e^2/T_{\parallel}$ ). The criterion for weak correlation (in the strongly magnetized case) can be written as  $\bar{b} \ll n^{-1/3}$ , where  $n$  is the electron density. The collisions that are most effective in producing an exchange of parallel and perpendicular energy are characterized by impact parameters that are small compared to the interparticle spacing (i.e.,  $\rho \ll b \ll n^{-1/3}$ ). These collisions tend to be well separated binary interactions, and their effect on the electron velocity distribution can be treated with a Boltzmann-like collision operator.

An analysis based on this collision operator leads to the equilibration equation  $dT_{\perp}/dt = (T_{\parallel} - T_{\perp})n\bar{b}^2\bar{v}_{\parallel}I(\bar{\kappa})$ , where  $n\bar{b}^2\bar{v}_{\parallel}$  is very nearly the usual Coulomb collision frequency,  $\bar{\kappa}$  is the parameter  $\kappa = \Omega b/v_{\parallel}$  with  $v_{\parallel}$  replaced by  $\bar{v}_{\parallel}$  and  $b$  replaced by  $\bar{b}$  (i.e.,  $\bar{\kappa} = \sqrt{2m}e^2\Omega/T_{\parallel}^{3/2}$ ), and the function  $I(\bar{\kappa})$  is essentially the average of  $[\exp(-\Omega\tau)]^2$

$= \exp(-\pi\kappa)$  over the distribution of relative parallel velocities. It is the square of  $\exp(-\Omega\tau)$  that enters the average, since the sign of the energy exchange varies randomly from collision to collision.

Since  $\kappa$  varies inversely as the cube of  $v_{\parallel}$  (i.e.,  $\kappa = 4e^2\Omega/mv_{\parallel}^3$ ), the quantity  $\exp(-\pi\kappa)$  is a rapidly increasing function of  $v_{\parallel}$ . On the other hand, the distribution of relative parallel velocities, a Maxwellian, is a rapidly decreasing function of  $v_{\parallel}$ . Consequently, the integral expression for  $I(\bar{\kappa})$ , which is an average of  $\exp(-\pi\kappa)$  over the distribution, contains the product of two functions that compete at large  $v_{\parallel}$ . Physically, this is easy to understand: very few collisions involve large relative velocities, but the collisions that do involve large relative velocities are very effective in producing an exchange of parallel and perpendicular energy. A saddle point evaluation of the integral yields the large  $\bar{\kappa}$  asymptotic result  $I(\bar{\kappa}) = [(0.47)\bar{\kappa}^{-1/5}] \times \exp[-(2.04)\bar{\kappa}^{2/5}]$ .

The main point to note here is that the equilibration rate is exponentially small in  $\bar{\kappa}^{2/5}$  rather than in  $\bar{\kappa}$ . This distinction is important, since  $\bar{\kappa}^{2/5} \ll \bar{\kappa}$  for  $\bar{\kappa} \gg 1$ . In other words, the collisions that involve large relative velocities produce an equilibration rate which is faster than one might have guessed.

For a sufficiently strong magnetic field and a sufficiently low electron temperature, the electron cyclotron motion must be treated quantum mechanically. Such a treatment is necessary when the de Broglie wavelength is comparable to the Larmor radius (i.e.,  $\hbar/m\bar{v}_{\perp} \sim \bar{v}_{\perp}/\Omega$  or  $\hbar\Omega \sim m\bar{v}_{\perp}^2$ ). In Sec. III, a quantum mechanical analysis is carried out for the case where  $T_{\perp} \sim \hbar\Omega$  but  $T_{\parallel} \gg \hbar\Omega$ . This ordering of  $T_{\parallel}$  relative to  $T_{\perp}$  is motivated by a current series of experiments.<sup>2</sup> The quantum analysis follows the same outline as the classical analysis and begins with the consideration of an isolated collision between two electrons in a strong magnetic field. As one would expect,<sup>3</sup> the probability of transitions that involve an exchange of parallel and perpendicular energy is exponentially small. When the transition probability is used in conjunction with a quantum version of the Boltzmann-like operator, an equilibration rate is obtained that has essentially the same value as the classical rate.

Although the parameter regime of strong magnetization is quite unusual, it is realized in a current series of experiments.<sup>2</sup> For  $T_{\parallel} = T_{\perp} = T$ , the inequality defining strong magnetization can be written as  $T^{3/2} \ll 10^{-7} B$ , where  $T$  is in electron volts and  $B$  is in Gauss. Even for  $B$  as large as 100 kG, the inequality is satisfied only for electron thermal energies that would lead to the recombination of a neutral plasma (i.e.,  $T \ll 0.1$  eV). However, the experiments alluded to involve the magnetic confinement of a pure electron plasma, and recombination cannot occur for such a plasma. Moreover, there is an effort to cool such a plasma to the liquid and crystal states,<sup>2,4</sup> and the low temperatures required (cryogenic range) are such that the plasma enters the parameter regime of strong magnetization.

Also, the results of this paper may have implications for the cooling effort. A cooling mechanism such as cyclotron radiation (the current choice) reduces  $T_{\perp}$  but not  $T_{\parallel}$ , and the question of whether or not collisions can maintain equiparti-

tion of energy depends on a comparison of the radiation rate and the equilibration rate calculated here.

A caveat must be appended to the entire previous discussion. A complete dynamical solution of the  $N$ -electron problem is out of the question. We have simply focused attention on those pieces of the dynamics that we believe are most effective in producing an exchange of parallel and perpendicular energy, that is, on the close two-particle collisions. However, we have not calculated the remaining dynamics and shown that it produces negligible exchange of parallel and perpendicular energy. For example, one might worry that a high-frequency instability ( $\omega \gg \Omega$ ) would break the adiabatic invariant and allow  $T_{\parallel}$  and  $T_{\perp}$  to equilibrate on a collective time scale. The equilibration rate resulting from close two-particle collisions should be interpreted as a lower bound on the equilibration rate.

## II. CLASSICAL ANALYSIS

This section presents a calculation of the equilibration rate for the case where the electron dynamics may be treated by classical mechanics (i.e.,  $T_{\parallel}, T_{\perp} \gg \hbar\Omega$ ). The first step in the calculation is to analyze an isolated collision between two electrons that interact electrostatically in the presence of a strong and uniform magnetic field,  $\mathbf{B} = \hat{z}B$ . The equations of motion for the electrons are

$$\frac{d\mathbf{v}_1}{dt} + \Omega\mathbf{v}_1 \times \hat{z} = \frac{e^2}{m} \frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}, \quad (1)$$

$$\frac{d\mathbf{v}_2}{dt} + \Omega\mathbf{v}_2 \times \hat{z} = \frac{e^2}{m} \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}, \quad (2)$$

where  $\mathbf{r}_j$  and  $\mathbf{v}_j$  are the position and velocity of electron  $j$ . By adding and subtracting these equations, we obtain the two equations

$$\frac{d\mathbf{V}}{dt} + \Omega\mathbf{V} \times \hat{z} = 0, \quad (3)$$

$$\frac{d\mathbf{v}}{dt} + \Omega\mathbf{v} \times \hat{z} = \frac{e^2}{\mu} \frac{\mathbf{r}}{|\mathbf{r}|^3}, \quad (4)$$

where  $\mathbf{V} = d/dt (\mathbf{r}_1 + \mathbf{r}_2)/2$  is the velocity of the center of mass,  $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$  is the position of electron 2 relative to that of electron 1,  $\mathbf{v} = d/dt (\mathbf{r})$  is the relative velocity, and  $\mu = m/2$  is the reduced mass. The center of mass motion is equivalent to that of an electron in a uniform magnetic field, and the relative motion is equivalent to that of an electron in a uniform magnetic field and the field of a fixed charge. The solution for the center of mass motion is trivial, and, as we will now see, the solution for the relative motion is simplified by the existence of an adiabatic invariant.

Let  $(v_{\parallel}, v_{\perp})$  be the parallel and perpendicular components of the relative velocity before the interaction, and let  $b$  be the distance of closest approach (i.e.,  $|\mathbf{r}| > b$ ). The conditions for strong magnetization imply that the gradient scale for the interaction is weak in the accessible region (i.e.,  $|\mathbf{r}| > b \gg v_{\perp}/\Omega, v_{\parallel}/\Omega$ ); thus  $v_{\perp}^2(t)/B$  is an adiabatic invariant. Since  $B$  is a constant, one can say equivalently that  $v_{\perp}^2(t)$  is an adiabatic invariant. Finally, we use the fact that the perpendicular energy is an adiabatic invariant to write the distance of closest approach as  $b = e^2/(\mu v_{\parallel}^2/2)$ .

Note that  $v_{\perp}^2(t)/B$  is a new adiabatic invariant associat-

ed jointly with the two electrons; neither  $v_{1\perp}^2(t)/B$  nor  $v_{2\perp}^2(t)/B$  are valid adiabatic invariants. For example, in Eq. (1),  $\mathbf{r}_2(t)$  is a time-dependent function that varies at the cyclotron frequency, and this breaks the adiabatic invariant of electron 1 [i.e.,  $v_{1\perp}^2(t)/B \neq \text{const}$ ]. Likewise, the temporal variation of  $\mathbf{r}_1(t)$  breaks the adiabatic invariant of electron 2. By introducing the relative position and velocity (i.e.,  $\mathbf{r}$  and  $\mathbf{v}$ ), we have removed the explicit time dependence from the interaction and uncovered a new adiabatic invariant,  $v_{\perp}^2(t)/B$ .

From Eq. (3), one can see that  $V_1^2(t)$  is an exact constant of the motion; the relation  $mv_{1\perp}^2(t)/2 + mv_{2\perp}^2(t)/2 = \mu v_{\perp}^2(t)/2 + (2m)V_1^2(t)/2$  implies that the sum of the perpendicular kinetic energies for the two electrons is an alternative expression for the adiabatic invariant. In Appendix A, this expression is generalized to the case where many electrons interact simultaneously, that is, the quantity  $\Sigma_j mv_{j\perp}^2(t)/2$  is shown to be an adiabatic invariant.

An adiabatic invariant is not strictly conserved but suffers exponentially small changes. For the case of a weakly correlated plasma, we will argue that the overall invariant [i.e.,  $\Sigma_j mv_{j\perp}^2(t)/2$ ] suffers changes primarily through close two-particle collisions; therefore we calculate the change that occurs in  $\mu v_{\perp}^2(t)/2$  during a collision. From Eq. (4), it follows that

$$\frac{d}{dt} \frac{\mu v_{\perp}^2(t)}{2} = \frac{e^2 \mathbf{v}_{\perp}(t) \cdot \mathbf{r}_1(t)}{|\mathbf{r}(t)|^3}, \quad (5)$$

thus, the quantity  $\Delta(\mu v_{\perp}^2/2) = \mu v_{\perp}^2(\infty)/2 - \mu v_{\perp}^2(-\infty)/2$  is given by the time integral

$$\Delta\left(\frac{\mu v_{\perp}^2}{2}\right) = \int_{-\infty}^{+\infty} dt \frac{e^2 \mathbf{v}_{\perp}(t) \cdot \mathbf{r}_1(t)}{|\mathbf{r}(t)|^3}. \quad (6)$$

Following the usual practice in the theory of adiabatic invariants, we use the lowest-order orbits in evaluating the time integral, that is, we rewrite Eq. (6) as

$$\Delta\left(\frac{\mu v_{\perp}^2}{2}\right) \simeq e^2 v_{\perp} \rho \int_{-\infty}^{+\infty} \frac{dt \cos(\Omega t + \delta)}{[\rho^2 + z^2(t)]^{3/2}}, \quad (7)$$

where  $(\rho, z)$  is the guiding center approximation for  $(\mathbf{r}_1, z)$ ,  $\delta$  is a constant, and  $z(t)$  is determined by

$$\dot{z}^2 + \frac{2e^2/\mu}{[\rho^2 + z^2(t)]^{1/2}} = \dot{z}^2(t = -\infty) = v_{\parallel}^2. \quad (8)$$

The origin of time may be shifted simply by changing the value of the constant  $\delta$ ; so, without loss of generality, we choose the origin of time so that  $z^2(t)$  is an even function of  $t$ , that is, so that the electron either passes  $z = 0$  or reflects at the time  $t = 0$ . Equation (8) then reduces to the form

$$\Delta\left(\frac{\mu v_{\perp}^2}{2}\right) = e^2 v_{\perp} \rho \cos(\delta) \int_{-\infty}^{+\infty} \frac{dt \cos(\Omega t)}{[\rho^2 + z^2(t)]^{3/2}}. \quad (9)$$

In terms of the scaled variables

$$\xi = z/b, \quad \xi = v_{\parallel} t/b, \quad \eta = \rho/b, \quad \kappa = \Omega b/v_{\parallel}, \quad (10)$$

the time integral in Eq. (9) can be written as

$$\int_{-\infty}^{+\infty} \frac{dt \cos(\Omega t)}{[\rho^2 + z^2(t)]^{3/2}} = \frac{1}{v_{\parallel} b^2} \int_{-\infty}^{+\infty} \frac{d\xi \cos(\kappa \xi)}{[\eta^2 + \xi^2(\xi)]^{3/2}}, \quad (11)$$

and Eq. (7) can be rewritten as

$$\left(\frac{d\xi}{d\xi}\right)^2 = 1 - \frac{1}{[\eta^2 + \xi^2(\xi)]^{1/2}}. \quad (12)$$

Since  $\kappa \gg 1$ , the  $\xi$  integral in Eq. (11) involves the product of a rapidly oscillating function and a slowly varying function and turns out to be exponentially small. In Appendix B, the integral is evaluated by deforming the contour of integration into the complex  $\xi$  plane and is shown to be of the form

$$\int_{-\infty}^{+\infty} \frac{d\xi \cos(\kappa \xi)}{[\eta^2 + \xi^2(\xi)]^{3/2}} \equiv h(\kappa, \eta) e^{-\kappa g(\eta)}, \quad (13)$$

where  $h(\kappa, \eta)$  is neither exponentially small nor exponentially large in the range of  $\kappa$  and  $\eta$  of interest, and  $g(\eta)$  is given by

$$g(\eta) = \left| \int_1^{\eta} \frac{x^{3/2} dx}{\sqrt{(x-1)(\eta^2-x^2)}} \right|. \quad (14)$$

From the plot of  $g(\eta)$  in Fig. 1, one can see that  $g(\eta) \simeq \pi/2$  for  $\eta = \rho/b \ll 1$  and that  $g(\eta) \simeq \eta$  for  $\eta = \rho/b \gg 1$ . Thus, the  $\xi$  integral is of order  $\exp[-\pi \Omega b/2v_{\parallel}]$  for  $\rho \ll b$  and of order  $\exp[-\Omega \rho/v_{\parallel}]$  for  $\rho \gg b$ . These exponentials are each of the form  $\exp(-\Omega \tau)$ , where  $\tau$  characterizes the duration of the collision. Also, we note that the  $\xi$  integral is largest for collisions characterized by small impact parameter and large relative velocity.

Next we turn to the question of how such collisions act cumulatively to produce the relaxation of  $T_{\parallel}$  and  $T_{\perp}$  in a strongly magnetized plasma. For simplicity, we consider the case of a weakly correlated plasma, that is, a plasma in which  $e^2 n^{1/3} \ll T_{\parallel}$ , where  $n$  is the electron density. This inequality can be rewritten as the condition that  $\bar{b}$  is small compared to the mean interparticle spacing (i.e.,  $\bar{b} \ll n^{-1/3}$ ). One can easily verify that correlations are determined by  $T_{\parallel}$  in the strongly magnetized parameter regime.

We have just seen that for the class of two-particle collisions it is the close collisions (i.e.,  $|\mathbf{r}_1 - \mathbf{r}_2| \lesssim \bar{b}$ ) that are most effective in producing an exchange of parallel and perpendicular energy. If we add a third particle into the dynamics, we obtain a small perturbation on the close two-particle colli-

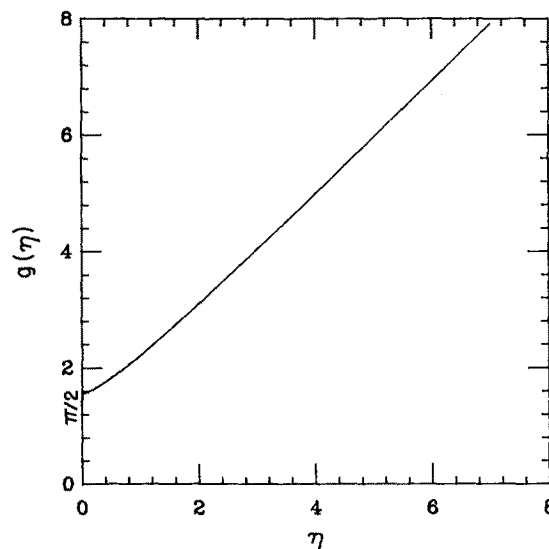


FIG. 1. A plot of the function  $g(\eta)$ , defined in Eq. (14).

sion unless all three particles are close simultaneously. We assume that the energy exchange caused by a close three-particle collision is of the same order as that for a close two-particle collision; this is reasonable considering time scale arguments concerning the durations of the collisions. Of course, we know from Appendix A that the overall adiabatic invariant,  $\Sigma_j m v_{j\perp}^2(t)/2$ , exists for many-electron collisions. Since the inequality  $\bar{b} \ll n^{-1/3}$  implies that close two-particle collisions are more frequent than close three-particle collisions, we neglect the close three-particle collisions and, similarly, all higher-order collisions. Also, we note that the close two-particle collisions are well-separated events (i.e.,  $|\mathbf{r}_1 - \mathbf{r}_2| \lesssim \bar{b} \ll n^{-1/3}$ ).

Such well-separated binary collisions can be treated with a Boltzmann-like collision operator.<sup>1,5</sup> In particular, we evaluate the integral

$$\frac{dT_{\perp}}{dt} = \int d\mathbf{v}_1 \frac{m v_{1\perp}^2}{2} \frac{\partial f}{\partial t}(\mathbf{v}_1, t) \quad (15)$$

by replacing the time derivative of the distribution function with the Boltzmann-like operator

$$\frac{\partial f}{\partial t}(\mathbf{v}_1, t) = n \int 2\pi\rho d\rho \int d\mathbf{v}_2 |\hat{z} \cdot (\mathbf{v}_2 - \mathbf{v}_1)| \times [f(\mathbf{v}'_1, t) f(\mathbf{v}'_2, t) - f(\mathbf{v}_1, t) f(\mathbf{v}_2, t)]. \quad (16)$$

This operator can be derived from the BBGKY hierarchy under the assumption that the plasma is strongly magnetized and that the most important collisions are well-separated binary interactions.<sup>1,5</sup> In relation to the usual form of the Boltzmann collision operator, the integral over  $2\pi\rho d\rho$  is equivalent to the integral over the impact parameter (or, scattering cross section), and the quantity  $|\hat{z} \cdot (\mathbf{v}_2 - \mathbf{v}_1)|$  replaces  $|\mathbf{v}_2 - \mathbf{v}_1|$ , since the two electrons stream toward one another along field lines. In the usual manner ( $\mathbf{v}'_1, \mathbf{v}'_2$ ) are velocities that evolve into  $(\mathbf{v}_1, \mathbf{v}_2)$  during a scattering.

By substituting the collision operator into Eq. (15) and by using detailed balance, we obtain

$$\begin{aligned} \frac{dT_{\perp}}{dt} &= \frac{n}{4} \int 2\pi\rho d\rho \int d\mathbf{v}_1 \int d\mathbf{v}_2 |\hat{z} \cdot (\mathbf{v}_2 - \mathbf{v}_1)| \\ &\times [f(\mathbf{v}'_1, t) f(\mathbf{v}'_2, t) - f(\mathbf{v}_1, t) f(\mathbf{v}_2, t)] \\ &\times \left( \frac{m v_{1\perp}^2}{2} + \frac{m v_{2\perp}^2}{2} - \frac{m v'_{1\perp}{}^2}{2} - \frac{m v'_{2\perp}{}^2}{2} \right). \end{aligned} \quad (17)$$

The distribution functions are assumed to be of the form

$$f(\mathbf{v}_j, t) = \left( \frac{m}{2\pi T_{\parallel}} \right)^{1/2} \left( \frac{m}{2\pi T_{\perp}} \right) \exp\left( -\frac{m v_{j\parallel}^2}{2T_{\parallel}} - \frac{m v_{j\perp}^2}{2T_{\perp}} \right). \quad (18)$$

To evaluate the multiple integral, it is convenient to introduce the center of mass velocity and the relative velocity (i.e.,  $\mathbf{V}$ , and  $\mathbf{v}$ ). With the aid of the relations

$$\begin{aligned} \frac{m v_{1\perp}^2}{2} + \frac{m v_{2\perp}^2}{2} &= \frac{\mu v_{\perp}^2}{2} + \frac{2m V_{\perp}^2}{2}, \\ \frac{m v_{1\parallel}^2}{2} + \frac{m v_{2\parallel}^2}{2} &= \frac{\mu v_{\parallel}^2}{2} + \frac{2m V_{\parallel}^2}{2}, \\ V_{\perp}^2 &= V_1^2, \quad d\mathbf{v}_1 d\mathbf{v}_2 = d\mathbf{v} d\mathbf{V}, \end{aligned} \quad (19)$$

Eq. (17) can be rewritten as

$$\begin{aligned} \frac{dT_{\perp}}{dt} &= \frac{n}{4} \int 2\pi\rho d\rho \int d\mathbf{v} |v_{\parallel}| [f_r(v_{\parallel}, v_{\perp}') - f_r(v_{\parallel}, v_{\perp})] \\ &\times \Delta(\mu v_{\perp}^2/2), \end{aligned} \quad (20)$$

where the integral over  $d\mathbf{V}$  has been evaluated and

$$f_r(v_{\parallel}, v_{\perp}) = \left( \frac{\mu}{2\pi T_{\parallel}} \right)^{1/2} \left( \frac{\mu}{2\pi T_{\perp}} \right) \exp\left( -\frac{\mu v_{\parallel}^2}{2T_{\parallel}} - \frac{\mu v_{\perp}^2}{2T_{\perp}} \right) \quad (21)$$

is the distribution of relative velocities.

By using  $\Delta(\mu v_{\perp}^2/2) = \mu v_{\perp}^2/2 - \mu v'_{\perp}{}^2/2 = \mu v_{\parallel}^2/2 - \mu v_{\parallel}'^2/2$ , Eq. (20) can be rewritten as

$$\begin{aligned} \frac{dT_{\perp}}{dt} &= \frac{n}{4} \int 2\pi\rho d\rho \int d\mathbf{v} |v_{\parallel}| f_r(v_{\parallel}, v_{\perp}) \Delta\left(\frac{\mu v_{\perp}^2}{2}\right) \\ &\times \left\{ \exp\left[ \left( \frac{1}{T_{\perp}} - \frac{1}{T_{\parallel}} \right) \Delta\left(\frac{\mu v_{\perp}^2}{2}\right) \right] - 1 \right\}. \end{aligned} \quad (22)$$

Taylor-expanding the exponential, substituting for  $\Delta(\mu v_{\perp}^2/2)$  from Eq. (9), and integrating over  $d\mathbf{v}_{\perp}$  yields the expression

$$\begin{aligned} \frac{dT_{\perp}}{dt} &= \left[ T_{\perp} \left( \frac{1}{T_{\perp}} - \frac{1}{T_{\parallel}} \right) \right] \frac{n}{4} \int 2\pi\rho d\rho \int d\mathbf{v}_{\parallel} |v_{\parallel}| \\ &\times \frac{\exp(-\mu v_{\parallel}^2/2T_{\parallel})}{(2\pi T_{\parallel}/\mu)^{1/2}} \frac{e^4 \rho^2}{\mu} \left( \int_{-\infty}^{+\infty} \frac{dt \cos(\Omega t)}{[\rho^2 + z^2(t)]^{3/2}} \right)^2. \end{aligned} \quad (23)$$

By using Eqs. (11) and (13) to reexpress the time integral, Eq. (23) reduces to the form

$$\begin{aligned} \frac{dT_{\perp}}{dt} &= (T_{\parallel} - T_{\perp}) n \bar{b}^2 \bar{v}_{\parallel} I(\bar{\kappa}), \\ I(\bar{\kappa}) &= \frac{\sqrt{2\pi}}{8} \int_0^{\infty} \frac{d\sigma}{\sigma} \exp\left( -\frac{\sigma^2}{2} \right) \\ &\times \int_0^{\infty} \eta^3 d\eta h^2\left( \frac{\bar{\kappa}}{\sigma^3}, \eta \right) \exp\left( -\frac{2\bar{\kappa}g(\eta)}{\sigma^3} \right), \end{aligned} \quad (24)$$

where we have introduced the scaled variables

$$\bar{v}_{\parallel} = \sqrt{T_{\parallel}}/\mu, \quad \sigma = v_{\parallel}/\bar{v}_{\parallel}, \quad \bar{b} = 2e^2/\mu\bar{v}_{\parallel}^2, \quad \bar{\kappa} = \Omega\bar{b}/\bar{v}_{\parallel}. \quad (25)$$

Since  $\bar{\kappa}$  is assumed to be large and  $g(\eta)$  is an increasing function of  $\eta$ , the main contribution to the  $\eta$  integral comes from small  $\eta$ . By making the approximations (see Appendix B)

$$\begin{aligned} g(\eta) &\simeq g(0) + g''(0)\eta^2/2 = 1.57 + (0.675)\eta^2, \\ h(\bar{\kappa}/\sigma^3, \eta) &\simeq h(\bar{\kappa}/\sigma^3, 0) = (2.79)(\bar{\kappa}/\sigma^3), \end{aligned} \quad (26)$$

we obtain

$$I(\bar{\kappa}) \simeq (0.67) \int_0^{\infty} d\sigma \frac{e^{-\sigma^2/2}}{\sigma} e^{-(3.14)\bar{\kappa}/\sigma^2}. \quad (27)$$

For large  $\bar{\kappa}$ , the  $\sigma$  integral may be carried out by the saddle point method, and the result is

$$I(\bar{\kappa}) = (0.47)\bar{\kappa}^{-1/5} e^{-(2.04)\bar{\kappa}^{-2/5}}. \quad (28)$$

In Fig. 2, this large  $\bar{\kappa}$  asymptotic expression for  $I(\bar{\kappa})$  is compared to the results of a numerical evaluation of  $I(\bar{\kappa})$ . The curve is a plot of the asymptotic expression, and the points

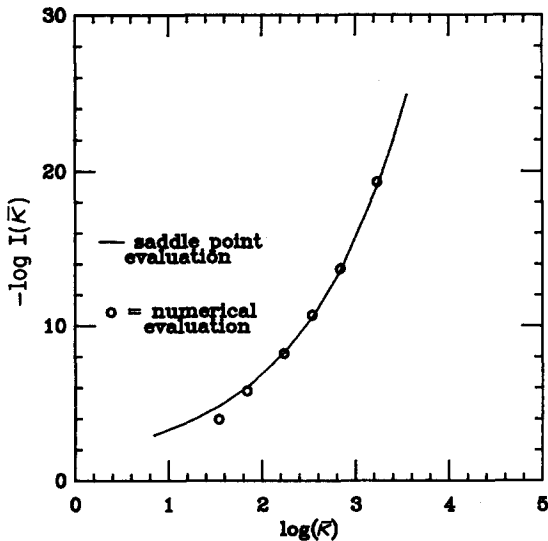


FIG. 2. The saddle point evaluation [Eq. (28)] of  $I(\bar{\kappa})$  compared to the result of a numerical evaluation of  $I(\bar{\kappa})$  for various values of  $\bar{\kappa}$ .

are the result of numerical integration for various values of  $\bar{\kappa}$ . One can see that the agreement is good for large enough values of  $\bar{\kappa}$ .

The main point to note here is that the equilibration rate is larger than one might have guessed. Since the exchange of parallel and perpendicular energy for an isolated collision between two electrons is exponentially small in  $\kappa$ , one might have guessed that the equilibration rate would be exponentially small in  $\bar{\kappa}$ . However, the equilibration rate turns out to be exponentially small in  $\bar{\kappa}^{2/5}$ , and this distinction is important since  $\bar{\kappa}^{2/5} \ll \bar{\kappa}$  for  $\bar{\kappa} > 1$ .

The  $\bar{\kappa}^{2/5}$  dependence is determined by a competition between the velocity dependence of  $\exp(-\pi\kappa) = \exp(-4\pi e^2\Omega/mv_{\parallel}^3)$  and the velocity dependence of the distribution of relative velocities,  $\exp(-\mu v_{\parallel}^2/2T_{\parallel})$ . Collisions characterized by large relative velocity are particularly effective at producing an exchange of parallel and perpendicular energy, but there are relatively few such collisions.

### III QUANTUM ANALYSIS

In this section, a quantum mechanical calculation of the equilibration rate is provided for the case where  $T_{\perp} \sim \hbar\Omega$  but  $T_{\parallel} \gg \hbar\Omega$ . This relative ordering of  $T_{\parallel}$  and  $T_{\perp}$  is motivated by the experiments mentioned in the introduction.<sup>2</sup> Since electron energies of the order  $\hbar\Omega$  are deemed relevant in this calculation, one should take into consideration the energy associated with orientation of the spins. Since the spin flip frequency is  $\Omega$ , the total spin action plus the total orbital cyclotron action form an adiabatic invariant. However, for the case of a uniform magnetic field, the spin dynamics and the orbital dynamics decouple, and the spins effectively drop out of the problem. Here, spin-spin interactions are neglected because the electron density is assumed to be very low.<sup>2</sup>

Following the same outline as the classical calculation, we begin by considering the scattering problem for two electrons that interact electrostatically in the presence of a

strong magnetic field. The Hamiltonian operator for these two electrons is given by

$$\hat{H} = \frac{[\hat{\mathbf{p}}_1 - (e/2c)\mathbf{r}_1 \times \mathbf{B}]^2}{2m} + \frac{[\hat{\mathbf{p}}_2 - (e/2c)\mathbf{r}_2 \times \mathbf{B}]^2}{2m} + e^2/|\mathbf{r}_2 - \mathbf{r}_1|, \quad (29)$$

where  $\hat{\mathbf{p}}_1 = (\hbar/i)\partial/\partial\mathbf{r}_1$  and  $\hat{\mathbf{p}}_2 = (\hbar/i)\partial/\partial\mathbf{r}_2$  are the momentum operators for electrons 1 and 2. In terms of the center of mass position,  $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ , and the relative position,  $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ , the Hamiltonian operator reduces to the form  $\hat{H} = \hat{H}_{\text{cm}} + \hat{H}_{\text{rel}}$ , where

$$\hat{H}_{\text{cm}} = [\hat{\mathbf{P}} - (e/c)\mathbf{R} \times \mathbf{B}]^2/4m, \quad (30)$$

$$\hat{H}_{\text{rel}} = [\hat{\mathbf{p}} - (e/4c)\mathbf{r} \times \mathbf{B}]^2/2\mu + e^2/|r|.$$

Here,  $\hat{\mathbf{P}} = (\hbar/i)\partial/\partial\mathbf{R}$  is the center of mass momentum operator and  $\hat{\mathbf{p}} = (\hbar/i)\partial/\partial\mathbf{r}$  is the relative momentum operator. Thus, the wavefunction can be expressed as the product  $\Psi = \Psi_{\text{cm}}(\mathbf{R}, t)\Psi_{\text{rel}}(\mathbf{r}, t)$ , where the evolution of  $\Psi_{\text{cm}}$  is determined by  $\hat{H}_{\text{cm}}$  and the evolution of  $\Psi_{\text{rel}}$  by  $\hat{H}_{\text{rel}}$ . Since  $\Psi_{\text{cm}}(\mathbf{R}, t)$  is not affected by the interaction, we need only consider the evolution of  $\Psi_{\text{rel}}$ , that is, the solution of the Schrödinger equation  $i\hbar\partial\Psi_{\text{rel}}/\partial t = \hat{H}_{\text{rel}}\Psi_{\text{rel}}$ .

In terms of cylindrical coordinates, the operator  $\hat{H}_{\text{rel}}$  takes the form

$$\hat{H}_{\text{rel}} = -\frac{\hbar^2}{2\mu} \left( \frac{\partial^2}{\partial z^2} + \frac{1}{r_1} \frac{\partial}{\partial r_1} r_1 \frac{\partial}{\partial r_1} + \frac{1}{r_1^2} \frac{\partial^2}{\partial \theta^2} \right) + \frac{\hbar\Omega}{2i} \frac{\partial}{\partial \theta} + \frac{\mu\Omega^2}{8} r_1^2 + \frac{e^2}{(r_1^2 + z^2)^{1/2}}. \quad (31)$$

Since  $\hat{p}_{\theta} = (\hbar/i)\partial/\partial\theta$  commutes with  $\hat{H}_{\text{rel}}$ , the angular momentum is a good quantum number, and we can look for wavefunctions of the form  $\Psi_{\text{rel}}(z, r_1, \theta, t) = \Psi_l(z, r_1, t) \exp(il\theta)$ . Replacing  $\partial/\partial\theta$  by  $il$  in  $\hat{H}_{\text{rel}}$  yields the reduced operator

$$\hat{H}_{\text{rel},l} = -\frac{\hbar^2}{2\mu} \left( \frac{\partial^2}{\partial z^2} + \frac{1}{r_1} \frac{\partial}{\partial r_1} r_1 \frac{\partial}{\partial r_1} \right) + \frac{\mu\Omega^2}{8r_1^2} (r_1^2 - \rho_l^2)^2 + e^2/(r_1^2 + z^2)^{1/2}, \quad (32)$$

where  $\rho_l$  is defined through the relation  $\hbar l = p_{\theta} = -\mu\Omega\rho_l^2/2$ .

We will find that the main contribution to the integral expression for the equilibration rate comes from wavefunctions representing electrons with perpendicular kinetic energy of order  $\hbar\Omega$  and with  $\rho_l$  of order  $b$ . These wavefunctions have large and negative values of the angular momentum quantum number:  $-l = \mu\Omega\rho_l^2/2\hbar \approx (\rho_l/r_L)^2 \gg 1$ , where  $r_L = (2\hbar/\mu\Omega)^{1/2}$  is the effective Larmor radius. The classical picture that corresponds to these wavefunctions is that of a reduced mass electron incident on the force center with an impact parameter  $r_1 = \rho_l$  and with a Larmor radius that is small compared to  $\rho_l$ . For the quantum case, the  $\theta$  location of the incident electron is completely uncertain, since the probability density is distributed uniformly in  $\theta$ .

Since the wavefunctions are peaked near  $r_1 = \rho_l$ , we set  $r_1 = \rho_l + x$  and Taylor-expand terms in  $\hat{H}_{\text{rel},l}$  with respect to  $x$ . To lowest nontrivial order,  $\hat{H}_{\text{rel},l}$  can be written as

$\hat{H}_{rel,i} = H_z^{(0)} + H_x^{(0)} + H^{(1)}$ , where

$$\begin{aligned} H_z^{(0)} &= -\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial z^2} + \frac{e^2}{(\rho_i^2 + z^2)^{1/2}}, \\ H_x^{(0)} &= -\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial x^2} + \frac{\mu\Omega^2}{2} x^2, \\ H^{(1)} &= -x\rho_i e^2 / (\rho_i^2 + z^2)^{3/2}. \end{aligned} \quad (33)$$

In zero order, the  $x$  and  $z$  dynamics decouple; so the zero-order eigenfunctions can be written as the product  $\psi_{v,\epsilon}(x,z) = G_v(x)F_\epsilon(z)$ , where  $\psi_{v,\epsilon}$ ,  $G_v$ , and  $F_\epsilon$  are solutions of the eigenvalue problems

$$\begin{aligned} (H_x^{(0)} + H_z^{(0)})\psi_{v,\epsilon}(x,z) &= E_{v,\epsilon}\psi_{v,\epsilon}(x,z), \\ H_x^{(0)} G_v(x) &= \hbar\Omega(v + \frac{1}{2})G_v(x), \\ H_z^{(0)} F_\epsilon(z) &= \epsilon F_\epsilon(z), \end{aligned} \quad (34)$$

and  $E_{v,\epsilon} = \hbar\Omega(v + \frac{1}{2}) + \epsilon$ . The functions  $G_v(x)$  are the well-known eigenfunctions for a harmonic oscillator,<sup>6</sup> and the functions  $F_\epsilon(z)$  are eigenfunctions for one-dimensional motion of energy  $\epsilon$  in the potential  $e^2/(\rho_i^2 + z^2)^{1/2}$ .

The perpendicular dynamics tends to be quantum mechanical in that small  $\nu$  values play an important role in the theory; recall that  $T_\perp \sim \hbar\Omega$ . On the other hand, the parallel dynamics tends to be quasiclassical in that the WKB method provides a useful approximation for  $F_\epsilon(z)$ . To see that this is the case, first note that  $b = e^2/\epsilon$  is the minimum scale length on which the potential varies in the classically accessible region. The condition that the de Broglie wavelength  $\lambda = \hbar/\sqrt{m\epsilon}$  be small compared to  $b$  can be rewritten as the inequality  $\epsilon \ll E_R$ , where  $E_R = \mu e^4/\hbar^2$  is the Rydberg energy (i.e., 13.6 eV). This inequality is easily satisfied for the low electron energies we have in mind (i.e.,  $T_\parallel \ll E_R$ ).

Since the  $x$  and  $z$  dynamics decouple in zero order, there is no energy exchange in zero order. The energy exchange is introduced by  $H^{(1)}(x,z)$ . According to Fermi's "golden rule," the probability per unit time of a transition from the state  $\psi_{v,\epsilon}$  to the state  $\psi_{v',\epsilon'}$  is given by

$$\frac{\text{probability}}{\text{time}} = \frac{2\pi}{\hbar} \frac{dn}{d\epsilon} |\langle \psi_{v',\epsilon'} | H^{(1)} | \psi_{v,\epsilon} \rangle|^2, \quad (35)$$

where  $dn/d\epsilon$  is the density of final states and it is assumed that the transition conserves the total energy (i.e.,  $\epsilon' + \hbar\Omega\nu' = \epsilon + \hbar\Omega\nu$ ).

In the Boltzmann analysis that follows, we will need the transition probability per collision, rather than the transition probability per unit time. Let  $J_{\epsilon'}$  be the incident flux associated with the wavefunction  $\psi_{v',\epsilon'}$ . Since  $J_{\epsilon'}$  is equivalently the number of collisions per unit time, the transition probability per collision is given by

$$P(\rho_i; \epsilon', \nu' \rightarrow \epsilon, \nu) = \left( \frac{\text{collisions}}{\text{time}} \right)^{-1} \left( \frac{\text{probability}}{\text{time}} \right), \quad (36)$$

$$P(\rho_i; \epsilon', \nu' \rightarrow \epsilon, \nu) = \frac{2\pi}{\hbar} (J_{\epsilon'})^{-1} \frac{dn}{d\epsilon} |\langle \psi_{v',\epsilon'} | H^{(1)} | \psi_{v,\epsilon} \rangle|^2.$$

To make  $(J_{\epsilon'})^{-1}$  and  $dn/d\epsilon$  finite (and the amplitude of  $\psi_{v,\epsilon}$  nonzero), we impose periodic boundary conditions at  $z = \pm L$ , where  $L$  is large (i.e.,  $L \gg \rho_i$ ) but finite. Over most of the  $z$  domain,  $F_\epsilon(z)$  is simply a free-space wavefunction; so a

discrete spectrum of energy values is introduced through the relation  $2kL = 2\pi n$ , where  $n$  is an integer and  $\hbar^2 k^2 / 2\mu = \epsilon$ . The density of states is given by  $dn/d\epsilon = (L/\pi) dk/d\epsilon = (1/2\pi\hbar)(2L/v_\parallel)$ , where  $v_\parallel = \hbar k/\mu$ . Likewise, the incident flux is given by  $J_{\epsilon'} = (v_\parallel/2L)$ . Thus, Eq. (36) reduces to the form

$$P(\rho_i; \epsilon', \nu' \rightarrow \epsilon, \nu) = (1/\hbar^2)(2L/v_\parallel)(2L/v'_\parallel) |\langle \psi_{v',\epsilon'} | H^{(1)} | \psi_{v,\epsilon} \rangle|^2. \quad (37)$$

The matrix element  $\langle \psi_{v',\epsilon'} | H^{(1)} | \psi_{v,\epsilon} \rangle$  is given by

$$\begin{aligned} \langle \psi_{v',\epsilon'} | H^{(1)} | \psi_{v,\epsilon} \rangle &= -e^2 \rho_i (\hbar/2\mu\Omega)^{1/2} (\sqrt{\nu} \delta'_{v,\nu-1} + \sqrt{\nu+1} \delta_{v,\nu+1}) \\ &\times \int_{-L}^L dz \frac{F_{\epsilon'}^*(z) F_\epsilon(z)}{(\rho_i^2 + z^2)^{3/2}}, \end{aligned} \quad (38)$$

where the integral  $\int dx G_v^*(x) x G_{v'}(x)$  has been evaluated using well-known properties of the harmonic oscillator eigenfunctions.<sup>6</sup> One can see that the allowed transitions involve an exchange of parallel and perpendicular energy of the amount  $\epsilon' - \epsilon = \hbar\Omega(\nu - \nu') = \pm \hbar\Omega$ . Substituting Eq. (38) into Eq. (37) yields the result

$$\begin{aligned} P(\rho_i; \epsilon', \nu' \rightarrow \epsilon, \nu) &= (e^4 \rho_i^2 / 2\mu \hbar \Omega) (\sqrt{\nu} \delta_{v,\nu-1} + \sqrt{\nu+1} \delta_{v,\nu+1})^2 \\ &\times \left| \frac{2L}{\sqrt{v'_\parallel v_\parallel}} \int_{-L}^L dz \frac{F_{\epsilon'}^*(z) F_\epsilon(z)}{(\rho_i^2 + z^2)^{3/2}} \right|^2. \end{aligned} \quad (39)$$

The  $z$  integral in this expression is most easily evaluated for the case of large parallel energy (i.e.,  $\epsilon, \epsilon' \sim T_\parallel \gg \hbar\Omega$ ) and large impact parameter (i.e.,  $\rho_i \gg b, b'$  or  $\epsilon, \epsilon' \gg e^2/\rho_i$ ). For this case, the probability of reflection from the potential  $e^2/(\rho_i^2 + z^2)^{1/2}$  is negligible, and the WKB solution for  $F_\epsilon(z)$  is of the form

$$F_\epsilon(z) = A/\sqrt{k(z)} \exp\left(i \int_0^z k(z) dz\right), \quad (40)$$

where  $\hbar^2 k^2(z)/2\mu + e^2/(\rho_i^2 + z^2)^{1/2} = \epsilon$ , and  $A$  is determined by the normalization  $1 = \int_{-L}^L dz |F_\epsilon(z)|^2 \simeq 2L |A|^2/k$ . The solution for  $F_{\epsilon'}(z)$  is obtained simply by replacing unprimed by primed quantities.

In terms of these WKB solutions, the  $z$  integral takes the form

$$\begin{aligned} \frac{2L}{\sqrt{v'_\parallel v_\parallel}} \int_{-L}^L dz \frac{F_{\epsilon'}^*(z) F_\epsilon(z)}{(\rho_i^2 + z^2)^{3/2}} &= \int_{-L}^L \frac{dz}{\sqrt{v'_\parallel(z) v_\parallel(z)}} \exp\left(i \int_0^z dz [k(z) - k'(z)]\right), \end{aligned} \quad (41)$$

where  $v_\parallel(z) = \hbar k(z)/\mu$ . By using the equations for  $k^2(z)$  and  $k'^2(z)$  and the relation  $\epsilon' - \epsilon = \pm \hbar\Omega$ , the wavenumber difference can be expressed as  $k'(z) - k(z) = \pm 2\Omega/[v'_\parallel(z) + v_\parallel(z)]$ . The integral can then be evaluated to lowest order in  $|k' - k|/k \sim \hbar\Omega/\epsilon \ll 1$  simply by setting  $v'_\parallel(z) = v_\parallel(z)$ . If we also introduce a time variable through the relation  $t = \int_0^z dz/v_\parallel(z)$ , the integral takes the form

$$\frac{2L}{\sqrt{v'_\parallel v_\parallel}} \int_{-L}^L dz \frac{F_{\epsilon'}^*(z) F_\epsilon(z)}{(\rho_i^2 + z^2)^{3/2}} = \int_{t(-L)}^{t(L)} dt \frac{\exp(\pm i\Omega t)}{[\rho_i^2 + z^2(t)]^{3/2}}, \quad (42)$$

and if we take into account the facts that  $z^2(t)$  is an even function of  $t$  and that  $L \gg \rho_i$ , the integral reduces to the familiar form

$$\frac{2L}{\sqrt{v_{\parallel} v'_{\parallel}}} \int_{-L}^L dz \frac{F_{\epsilon}^*(z) F_{\epsilon}(z)}{(\rho_i^2 + z^2)^{3/2}} \simeq \int_{-\infty}^{\infty} dt \frac{\cos(\Omega t)}{[\rho_i^2 + z^2(t)]^{3/2}}. \quad (43)$$

This time integral is the integral that enters the classical expression for the change in the adiabatic invariant [see Eq. (9)]; it is exponentially small.

We have been considering collisions that are characterized by large parallel energy (i.e.,  $\epsilon, \epsilon' \sim T_{\parallel} \gg \hbar\Omega$ ) and large impact parameter (i.e.,  $\epsilon, \epsilon' \gg e^2/\rho_i$ ). Equation (43) is most easily demonstrated for such collisions, but such collisions do not ultimately make the dominant contribution in the expression for the equilibration rate. As one would expect by analogy with the classical analysis, the dominant contribution is made by collisions characterized by large parallel energy (i.e.,  $\epsilon, \epsilon' \sim T_{\parallel} \gg \hbar\Omega$ ) and small impact parameter (i.e.,  $\epsilon, \epsilon' \ll e^2/\rho_i$ ). For this case, the solutions for  $F_{\epsilon}(z)$  and  $F_{\epsilon'}(z)$  encounter turning points near which the WKB approximation breaks down. However, in Appendix C, the  $z$  integral is analytically continued into the complex  $z$  plane, that is, away from the turning points on the real axis, and Eq. (43) is again obtained. In what follows, we use Eq. (43) as a general result and rewrite the transition probability as

$$P(\rho_i, \epsilon', \nu', \rightarrow \epsilon, \nu) = \left( \frac{e^4 \rho_i^2}{2 \mu \hbar \Omega} \right) (\sqrt{\nu} \delta_{\nu, \nu-1} + \sqrt{\nu+1} \delta_{\nu, \nu+1})^2 \times \left( \int_{-\infty}^{+\infty} dt \frac{\cos(\Omega t)}{[\rho_i^2 + z^2(t)]^{3/2}} \right)^2. \quad (44)$$

By analogy with Eq. (20), one can see that the rate of change of the perpendicular temperature is given by the expression

$$\frac{dT_{\perp}}{dt} = \frac{n}{4} \int 2\pi \rho_i d\rho_i \int dv |v_{\parallel}| \sum_{\nu, \nu'} P(\rho_i; \epsilon' \nu' \rightarrow \epsilon, \nu) \times [f(v'_{\parallel}, \nu') - f(v_{\parallel}, \nu)] \hbar\Omega(\nu - \nu'), \quad (45)$$

where the distribution of relative velocities is given by

$$f_r(v_{\parallel}, \nu) = \frac{2 \sinh(\hbar\Omega/2T_{\perp})}{(2\pi T_{\parallel}/\mu)^{1/2}} \exp\left(-\frac{\mu v_{\parallel}^2}{2T_{\parallel}} - \frac{\hbar\Omega(\nu + \frac{1}{2})}{T_{\perp}}\right). \quad (46)$$

By using the relations  $P(\rho_i; \epsilon', \nu', \rightarrow \epsilon, \nu) = P(\rho_i; \epsilon, \nu \rightarrow \epsilon', \nu')$  and  $\epsilon + \hbar\Omega\nu = \epsilon' + \hbar\Omega\nu'$ , Eq. (45) can be rewritten as

$$\frac{dT_{\perp}}{dt} = \frac{n}{2} \int 2\pi \rho_i d\rho_i \int dv |v_{\parallel}| \sum_{\nu < \nu'} P(\rho_i; \epsilon' \nu' \rightarrow \epsilon, \nu) \times f_r(v_{\parallel}, \nu) \left\{ 1 - \exp\left[\left(\frac{1}{T_{\parallel}} - \frac{1}{T_{\perp}}\right) \times \hbar\Omega(\nu' - \nu)\right] \right\} \hbar\Omega(\nu' - \nu), \quad (47)$$

which is manifestly positive for  $T_{\parallel} > T_{\perp}$ . Finally, substituting for  $P(\rho_i; \epsilon' \nu' \rightarrow \epsilon, \nu)$  yields the result

$$\frac{dT_{\perp}}{dt} = \left( \frac{1 - \exp[(1/T_{\parallel} - 1/T_{\perp})\hbar\Omega]}{1 - \exp(-\hbar\Omega/T_{\perp})} \right) \frac{n}{4} \times \int 2\pi \rho d\rho \int dv_{\parallel} |v_{\parallel}| \frac{\exp(-\mu v_{\parallel}^2/2T_{\parallel})}{(2\pi T_{\parallel}/\mu)^{1/2}} \frac{e^4 \rho_i^2}{\mu} \times \left( \int_{-\infty}^{\infty} \frac{dt \cos(\Omega t)}{[\rho_i^2 + z^2(t)]^{3/2}} \right)^2, \quad (48)$$

where use has been made of the sum

$$\sum_{\nu=0}^{\infty} (\nu+1) \exp\left(-\frac{\hbar\Omega(\nu + \frac{1}{2})}{T_{\perp}}\right) = [2 \sinh(\hbar\Omega/2T_{\perp})]^{-1} [1 - \exp(-\hbar\Omega/T_{\perp})]^{-1}. \quad (49)$$

By comparison with Eq. (23) of the previous section, one can see that Eq. (48) differs from the classical result only in that the large parentheses in front have replaced the bracket  $[T_{\perp}(1/T_{\perp} - 1/T_{\parallel})]$ . This is consistent with letting  $\hbar$  go to zero in Eq. (48). One can also see that the classical result and the quantum result are the same for the ordering considered in this section (i.e.,  $T_{\perp} \sim \hbar\Omega$  but  $T_{\parallel} \gg \hbar\Omega$ ). In particular, the square bracket in Eq. (48) is nearly unity provided that  $T_{\parallel} \gg \hbar\Omega$ , regardless of the relative ordering of  $T_{\perp}$  and  $\hbar\Omega$ .

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## APPENDIX A: AN ALTERNATIVE DERIVATION OF THE ADIABATIC INVARIANT

In this appendix, we consider collisions involving the simultaneous interaction of many electrons. For each electron we use the canonical coordinates and momenta<sup>7</sup> ( $z, p_z, Y, m\Omega X, \psi, p_{\psi}$ ), where

$$\begin{aligned} \tan \psi &= -v_x/v_y, \\ p_{\psi} &= m(v_x^2 + v_y^2)/2\Omega, \\ X &= x - v_y/\Omega, \quad Y = y + v_x/\Omega. \end{aligned} \quad (A1)$$

Here,  $\psi$  is the gyroangle and  $p_{\psi}$  is its conjugate momentum, and  $(X, Y)$  are the guiding center coordinates,  $P_Y = m\Omega X$  being the momentum conjugate to  $Y$ . The Hamiltonian for the electrons is given by

$$H = \sum_{j=1}^N \frac{p_{z_j}^2}{2m} + \Omega p_{\psi_j} + \sum_{i < j} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|}, \quad (A2)$$

where

$$\begin{aligned} |\mathbf{r}_i - \mathbf{r}_j|^2 &= (X_i + \rho_i \cos \psi_i - X_j - \rho_j \cos \psi_j)^2 \\ &\quad + (Y_i + \rho_i \sin \psi_i - Y_j - \rho_j \sin \psi_j)^2 \\ &\quad + (z_i - z_j)^2. \end{aligned} \quad (A3)$$

The quantity  $\rho = (2p_{\psi}/m\Omega)^{1/2}$  is the Larmor radius for an electron.

Assuming that the dynamics is that of a many-electron collision, rather than a collective mode of oscillation, the

inequality  $\Omega \gg v_{\perp}/b, v_{\parallel}/b$  implies that the  $\psi_j$  are rapidly varying (i.e.,  $\dot{\psi}_j = \partial H / \partial p_{\psi_j} \simeq \Omega$ ) compared to the other variables. Since there are many fast variables (i.e.,  $\psi_j$ , for  $j = 1, \dots, N$ ), the existence of an adiabatic invariant is not immediately obvious.

To uncover the invariant, we make a transformation to a new set of variables such that only one of the variables is rapidly varying. The transformation takes  $\{(\psi_j, p_{\psi_j}) | j = 1, \dots, N\}$  into  $\{(\theta_j, p_{\theta_j}) | j = 1, \dots, N\}$  via the generating function<sup>8</sup>

$$F_2 = p_{\theta_1} \psi_1 + \sum_{j=2}^N p_{\theta_j} (\psi_j - \psi_1) \quad (\text{A4})$$

and leaves the variables  $(z_j, p_z, Y_j, m\Omega Y_j)$  unchanged. Of course, an identity transformation for these latter variables could have been added to the generating function. The new variables are related to the old by taking partial derivatives in the usual manner<sup>8</sup>:

$$\theta_1 = \frac{\partial F_2}{\partial p_{\theta_1}} = \psi_1, \quad \theta_j = \frac{\partial F_2}{\partial p_{\theta_j}} = \psi_j - \psi_1, \quad \text{for } j > 1, \quad (\text{A5})$$

$$p_{\psi_1} = \frac{\partial F_2}{\partial \psi_1} = p_{\theta_1} - \sum_{j=2}^N p_{\theta_j}, \quad p_{\psi_j} = \frac{\partial F_2}{\partial \psi_j} = p_{\theta_j}, \quad \text{for } j \geq 1. \quad (\text{A6})$$

From Eq. (A6), it follows that  $p_{\theta_1} = \sum_{j=1}^N p_{\psi_j}$ ; so the Hamiltonian takes the form

$$H = p_{\theta_1} \Omega + \sum_{j=1}^N \frac{p_{\psi_j}^2}{2m} + \sum_{i < j} \frac{e^2}{|r_i - r_j|}. \quad (\text{A7})$$

For the Hamiltonian, one can see that  $\theta_1$  is the only rapidly varying variable. Also, when the slow variation is suppressed,  $(\theta_1, p_{\theta_1})$  are action angle variables. Thus,  $p_{\theta_1} = \sum_{j=1}^N p_{\psi_j}$  is an adiabatic invariant. From the definition of  $p_{\psi_j}$  in Eq. (A1), one can see that the adiabatic invariant is the total perpendicular action associated with the cyclotron motion (i.e.,  $p_{\theta_1} = \sum_{j=1}^N m v_{\perp j}^2 / 2\Omega$ ), and for the case of uniform magnetic field, this reduces to the total perpendicular kinetic energy.

## APPENDIX B: EVALUATION OF THE OSCILLATORY INTEGRAL IN EQ. (11)

In Sec. II, we found that the change in the adiabatic invariant is proportional to the oscillatory integral

$$\int_{-\infty}^{+\infty} \frac{d\xi \cos(\kappa\xi)}{[\eta^2 + \xi^2(\xi)]^{3/2}}, \quad (\text{B1})$$

where  $\kappa \gg 1$  and  $\xi(\xi)$  satisfies the differential equation

$$\left(\frac{d\xi}{d\xi}\right)^2 + [\eta^2 + \xi^2(\xi)]^{-1/2} = 1. \quad (\text{B2})$$

The choice of initial condition is that  $\xi^2(\xi)$  take its minimum value for  $\xi = 0$ , and, for this choice,  $\xi^2(\xi)$  is an even function of  $\xi$ . First, let us examine the integral in the two limiting cases  $\eta \gg 1$  and  $\eta \ll 1$ .

For  $\eta \gg 1$ , the solution of Eq. (B2) that satisfies the condition  $\xi^2(0) = 0$  is approximately  $\xi = \xi$ . This solution describes the motion of an electron that is incident at such large impact parameter that the interaction potential does not sig-

nificantly alter the electron motion. For this solution, integral (B1) is a representation of the modified Bessel function  $K_1(\kappa\eta)$ ,

$$\int_{-\infty}^{+\infty} \frac{d\xi \cos(\kappa\xi)}{[\eta^2 + \xi^2(\xi)]^{3/2}} = \frac{2\kappa}{\eta} K_1(\kappa\eta) \simeq \left(\frac{2\pi\kappa}{\eta^3}\right)^{3/2} e^{-\kappa\eta}. \quad (\text{B3})$$

As expected, the integral is exponentially small, and  $\kappa$  appears as a factor in the exponent.

The limit  $\eta \ll 1$  corresponds physically to a collision that is nearly head on. For such a collision, the sign of  $\xi(\xi)$  does not change, and to be specific we choose  $\xi > 0$ . Equation (B2) then reduces to the form

$$\left(\frac{d\xi}{d\xi}\right)^2 \simeq \frac{\xi - 1}{\xi}, \quad (\text{B4})$$

and the solution has the following character. As  $\xi$  varies from  $-\infty$  to  $+\infty$ ,  $\xi(\xi)$  varies from  $+\infty$  to 1 (at  $\xi = 0$ ) and then back to  $+\infty$ , with  $d\xi/d\xi$  changing sign at the turning point (i.e., at  $\xi = 1$ ).

This can be represented in the complex  $\xi$  plane as the motion of  $\xi(\xi)$  in along one side of the branch cut for  $\sqrt{\xi - 1}$  and then back out along the other. We take the square root of both sides of Eq. (B4) and choose the branch cut for  $\sqrt{\xi - 1}$  along the line  $\arg(\xi - 1) = 0$  and the branch cut for  $\sqrt{\xi}$  along the line  $\arg(\xi) = \pi$  [see Fig. 3(b)]. As  $\xi$  varies along the dashed contour in Fig. 3(a),  $\xi(\xi)$  varies along the dashed contour in Fig. 3(b).

Bearing in mind that in integral (B1) the factor  $\cos(\kappa\xi)$  may be replaced by  $\exp(i\kappa\xi)$ , since  $\xi^2(\xi)$  is even, we deform the  $\xi$  and  $\xi$  contours in such a way that  $\xi$  takes positive imaginary values. The deformation may be pushed as an analytic continuation process until the  $\xi$  contour collides with the branch cut associated with  $\sqrt{\xi}$ . Figures 4(a) and 4(b) show the  $\xi$  and  $\xi$  contours when the  $\xi$  contour has been pushed as far as possible toward positive imaginary  $\xi$ . The analytic continuation process yields an alternate representation for integral (B1):

$$\int_{-\infty}^{+\infty} d\xi \frac{e^{i\kappa\xi}}{\xi^3(\xi)} = \int_C d\xi \frac{e^{i\kappa\xi}}{\xi^3(\xi)}, \quad (\text{B5})$$

where  $C$  is the dashed contour in Fig. 4(a).

During the deformation, the turning point moves from  $\xi = 1$  to  $\xi = 0$ , and, according to Eq. (B4), the image of this point in the  $\xi$  plane moves from  $\xi = 0$  to

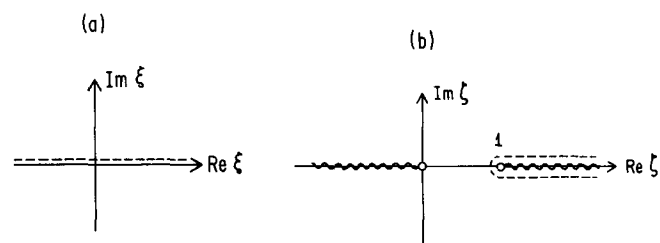


FIG. 3. Consistent contours in (a) the complex  $\xi$  plane and (b) the complex  $\xi$  plane for the motion  $\xi(\xi)$  of Eq. (B2) in the limit  $\eta \ll 1$ .



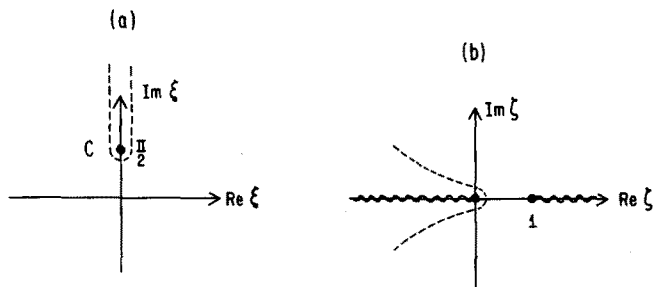


FIG. 4. The contours of Fig. 3 analytically continued as far as possible toward positive imaginary values of  $\xi$ .

$$\xi = \int_0^1 d\xi \frac{\sqrt{\xi}}{\sqrt{1-\xi}} = \frac{i\pi}{2}. \quad (B6)$$

Contour  $C$  loops down around this point. We can solve Eq. (B4) in the neighborhood of this point to obtain  $\xi^{3/2} = i(\xi - i\pi/2)$  or  $\xi^3 = -2(\xi - i\pi/2)^2$ . The integrand of integral (B5) has a second-order pole at  $\xi = i\pi/2$ , and the residue at this pole yields the result

$$\int_C d\xi \frac{e^{i\kappa\xi}}{\xi^3(\xi)} = \frac{8\pi\kappa}{9} e^{-(\pi/2)\kappa}. \quad (B7)$$

Thus, we find for both  $\eta > 1$  and  $\eta < 1$  that integral (B1) is of the form  $h(\kappa, \eta) \exp[-\kappa g(\eta)]$ , where  $h(\kappa, \eta)$  is nonexponential, and the function  $g(\eta)$  takes the limiting forms  $g(\eta) \simeq \pi/2$  for  $\eta < 1$  and  $g(\eta) \simeq \eta$  for  $\eta > 1$ . We now generalize this result to arbitrary values of  $\eta < 1$ , that is, to all cases where there is reflection. The cases where there is no reflection (i.e.,  $\eta > 1$ ) follow similarly.

It is useful to introduce the variable  $x = \sqrt{\xi^2 + \eta^2}$ , which satisfies the differential equation

$$\frac{dx}{d\xi} = \frac{i\sqrt{x^2 - \eta^2} \sqrt{x-1}}{x\sqrt{-x}}, \quad (B8)$$

where the branch cut for  $\sqrt{w(x)}$  is taken along  $\arg w(x) = 0$  and  $w(x) = x^2 - \eta^2, x - 1, -x$ . As  $\xi$  varies from  $-\infty$  to  $+\infty$  along the dashed contour in Fig. 5(a),  $x(\xi)$  varies along the dashed contour in Fig. 5(b), reaching the turning point  $x = 1$  for  $\xi = 0$  [i.e.,  $x(0) = 1$ ]. We deform these two contours so that  $\xi$  moves toward positive imaginary values, continuing the deformation process until the  $x$  contour collides with the branch point at  $x = \eta$ . This yields  $\xi$  and  $x$  contours of the form shown in Fig. 6.

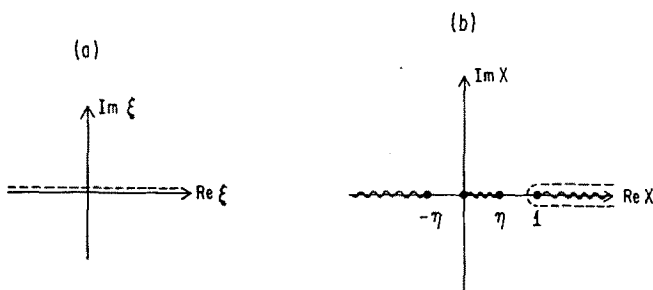


FIG. 5. Consistent contours in (a) the complex  $\xi$  plane and (b) the complex  $x$  plane for the motion  $x(\xi)$  of Eq. (B8) in the case  $0 < \eta < 1$ .

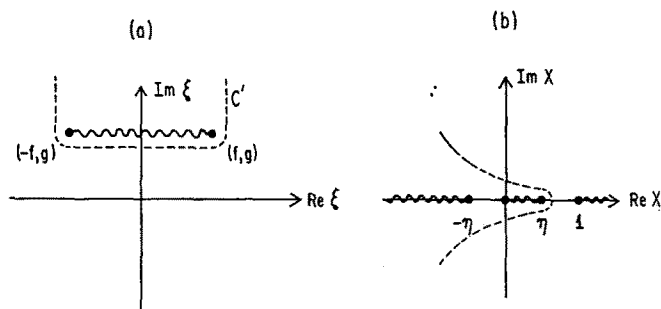


FIG. 6. The contours of Fig. 5 analytically continued as far as possible toward positive imaginary values of  $\xi$ .

During the deformation, the turning point moves from  $x = 1$  to  $x = \eta$ , and according to Eq. (B8), the image of the turning point moves from  $\xi = 0$  to

$$\xi = ig(\eta) = i \int_{\eta}^1 \frac{x^{3/2} dx}{\sqrt{1-x} \sqrt{x^2 - \eta^2}}. \quad (B9)$$

The two points around which the  $\xi$  contour loops are the images of  $x = 0$  approached from opposite sides of the branch cut between  $x = 0$  and  $x = \eta$ . The coordinates of these two points are  $\xi = ig(\eta) \pm f(\eta)$ , where

$$f(\eta) = \int_0^{\eta} \frac{x^{3/2} dx}{\sqrt{1-x} \sqrt{\eta^2 - x^2}}, \quad (B10)$$

and there is a branch cut of  $x(\xi)$  along the line between the two points. Integral (B1) can be rewritten as

$$\int_{C'} d\xi \frac{\exp(i\kappa\xi)}{x^3(\xi)}, \quad (B11)$$

where  $C'$  is the contour in Fig. 6(a). Since the singularities of the integral involve more than isolated poles, the integral cannot be evaluated simply as a sum of residues. Nevertheless, one can see that the integral is of order  $\exp[-\kappa g(\eta)]$ . The same conclusion is reached for the case where  $\eta > 1$ .

The value of the integral is determined numerically. Mutually consistent contours of the form shown in Fig. 6 are established numerically, and the integral is evaluated numerically along the  $\xi$  contour (i.e., along  $C'$ ). By repeating this procedure for an array of  $(\kappa, \eta)$  values, one can verify that the integral is of the form  $h(\kappa, \eta) \exp[-\kappa g(\eta)]$ , where  $h(\kappa, \eta)$  is not exponentially large or small in the parameter range of interest. One can also verify the small  $\eta$  expansions used in Eq. (26).

Also, the expression for  $I(\bar{\kappa})$  in Eq. (24) was evaluated numerically. This involves an integration over  $(\sigma, \eta)$  space, where  $\kappa = \bar{\kappa}/\sigma^3$ . However, because of the exponential character of the combined integrand, only values of  $(\sigma, \eta)$  in a neighborhood around some critical point  $(\sigma_0, \eta_0)$  contribute to the integral, the integrand falling exponentially to 0 away from  $(\sigma_0, \eta_0)$ ; see Fig. 7. Once  $(\sigma_0, \eta_0)$  is established, a suitable number ( $\approx 50$ ) of the values were chosen to represent the integral around that point, and the integrals over  $\sigma$  and  $\eta$  were then carried out. The results of these computations appear in Fig. 2, where the numerical data for the integral  $I(\bar{\kappa})$  are compared with the  $\bar{\kappa}$  asymptotic result of Sec. II. The agreement is reasonable for large  $\bar{\kappa}$ .

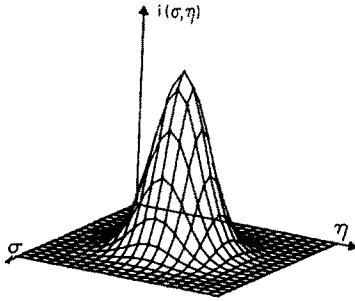


FIG. 7. The integrand of Eq. (24),  $i(\sigma, \eta)$ , falls exponentially to 0 away from a critical point  $(\sigma_0, \eta_0)$  in the  $(\sigma, \eta)$  plane.

### APPENDIX C: DISCUSSION OF EQ. (42)

In this appendix, we argue that collisions characterized by large parallel energy (i.e.,  $\epsilon, \epsilon' \gtrsim T_{\parallel} \gg \hbar\Omega$ ) and small impact parameter (i.e.,  $\rho_1 \ll b, b'$  or  $\epsilon, \epsilon' \ll e^2/\rho_1$ ) make the dominant contribution in the expression for the equilibration rate. This result is what one would expect by analogy with the classical analysis. Also, we verify Eq. (42) for such collisions.

Let us consider this latter point first. For the case of small impact parameter, the solution for  $F_{\epsilon}(z)$  encounters turning points at  $z = \pm a$ , where  $e^2/(\rho_1^2 + a^2)^{1/2} = \epsilon$ . Since  $a \simeq b$  is large, the probability of tunneling from  $a$  to  $-a$  is negligible (recall that  $\hbar/\sqrt{m\epsilon} \ll b$ ), and an appropriate WKB solution is of the form

$$F_{\epsilon}(z) = \begin{cases} \frac{2A}{\sqrt{k(z)}} \cos\left(\int_a^z k(z)dz - \frac{\pi}{4}\right), & \text{for } z > a, \\ \frac{A}{\sqrt{|k(z)|}} \exp\left(\int_a^z |k(z)|dz\right), & \text{for } z < a. \end{cases} \quad (\text{C1})$$

This corresponds to incidence from positive  $z$  and total reflection at the turning point; of course, the solution is not valid in the neighborhood of the turning point. The solution for  $F_{\epsilon'}(z)$  is of the same form as that for  $F_{\epsilon}(z)$ .

Following Landau's analysis of matrix elements in the quasiclassical limit,<sup>9</sup> we express  $F_{\epsilon}(z)$  as the sum of two WKB solutions:

$$F_{\epsilon}^{+}(z) = \begin{cases} \frac{A}{\sqrt{k(z)}} \exp\left(i \int_a^z k(z)dz - \frac{i\pi}{4}\right), & \text{for } z > a, \\ \frac{-iA}{\sqrt{|k(z)|}} \exp\left(-\int_a^z |k(z)|dz\right), & \text{for } z < a, \end{cases} \quad (\text{C2})$$

and

$$F_{\epsilon}^{-}(z) = \begin{cases} \frac{A}{\sqrt{k(z)}} \exp\left(-i \int_a^z k(z)dz + \frac{i\pi}{4}\right), & \text{for } z > a, \\ \frac{+iA}{\sqrt{|k(z)|}} \exp\left(-\int_a^z |k(z)|dz\right), & \text{for } z < a. \end{cases} \quad (\text{C3})$$

That  $F_{\epsilon}^{+}(z) + F_{\epsilon}^{-}(z)$  is a proper decomposition of  $F_{\epsilon}(z)$  requires some explanation. One must bear in mind that the expressions given for these quantities are not exact but are only the first terms in asymptotic series. For  $z > a$  it is clear that  $F_{\epsilon}^{+}(z) + F_{\epsilon}^{-}(z) = F_{\epsilon}(z)$ , but for  $z < a$  the expressions for

$F_{\epsilon}^{+}(z)$  and  $F_{\epsilon}^{-}(z)$  add to zero rather than to the exponentially small quantity  $F_{\epsilon}(z)$ . The true solutions for  $F_{\epsilon}^{+}(z)$  and for  $F_{\epsilon}^{-}(z)$  have additional terms, and it is these terms that add to give the exponentially small quantity  $F_{\epsilon}(z)$ .

It is useful to look at  $F_{\epsilon}(z)$ ,  $F_{\epsilon}^{+}(z)$ , and  $F_{\epsilon}^{-}(z)$  in the complex  $z$  plane. Near the turning point, the wavenumber is given by  $k(z) = \alpha\sqrt{z-a}$  and the integral of the wavenumber is given by  $\int_a^z k(z)dz = (2\alpha/3)(z-a)^{3/2}$ , where  $\alpha \simeq (2\mu\epsilon/\hbar^2 a)^{1/2}$  and we choose the branch cut along the line  $\arg(z-a) = 0$  (see Fig. 8). There are anti-Stokes lines at  $\arg(z-a) = 0, 2\pi/3, \text{ and } 4\pi/3$ , and we denote the regions bounded by these lines as regions I, II, and III. In regions I and III, the solution  $\exp[-i\int_a^z k(z)dz]$  is dominant, and in region II, the solution  $\exp[+i\int_a^z k(z)dz]$  is dominant. In general, the solution is the sum of a dominant and a subdominant solution, and, in analytically continuing through a given region, it is permissible to retain the subdominant solution only if the dominant solution enters with zero coefficient.

For example, for  $F_{\epsilon}^{+}(z)$  the dominant solution enters with zero coefficient in region I, and the solution

$$F_{\epsilon}^{+}(z) = \frac{A}{\sqrt{k(z)}} \exp\left(i \int_a^z k(z)dz - \frac{i\pi}{4}\right) \quad (\text{C4})$$

may be analytically continued through region I into region II. One may easily check that for  $\arg(z-a) = \pi$  this expression reproduces that given in Eq. (C2) for  $z < a$ . In fact, this analytic continuation is one method of verifying that the two expressions in Eq. (C2) match onto the same overall solution.

Since the solutions must be continuous across the branch cut, even though  $k(z)$  is discontinuous, we write  $F_{\epsilon}^{-}(z)$  as

$$F_{\epsilon}^{-}(z) = \frac{A}{\sqrt{k(z)}} \exp\left(+i \int_a^z k(z)dz + \frac{i3\pi}{4}\right), \quad (\text{C5})$$

for  $\arg(z-a) = 2\pi$ . Note that the expression in the first line of Eq. (C3) is for  $\arg(z-a) = 0$ . Since the dominant solution enters with zero coefficient in region III, expression (C5) may be analytically continued through this region into region II. One can easily check that expression (C5) reproduces the expression in the second line of Eq. (C3) when evaluated for  $\arg(z-a) = \pi$ .

Likewise, one can check that the function

$$F_{\epsilon}(z) = \frac{A}{\sqrt{k(z)}} \exp\left(-i \int_a^z k(z)dz + \frac{i\pi}{4}\right) \quad (\text{C6})$$

reproduces the second line of Eq. (C1) when evaluated for  $\arg(z-a) = \pi$ . This function is valid in the whole complex plane except near the positive real  $(z-a)$  axis.

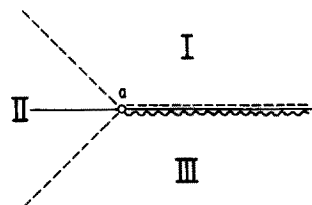


FIG. 8. Regions of the complex  $z$  plane in the vicinity of the turning point  $a$ . The dashed lines are anti-Stokes lines separating regions of dominance and subdominance for a WKB solution.

The  $z$  integral in Eq. (42) can be rewritten as

$$\int_C dz \frac{F_\epsilon(z)F_\epsilon(z)}{(\rho_i^2 + z^2)^{3/2}} = \int_C dz \frac{F_\epsilon(z)F_\epsilon^+(z)}{(\rho_i^2 + z^2)^{3/2}} + \int_C dz \frac{F_\epsilon(z)F_\epsilon^-(z)}{(\rho_i^2 + z^2)^{3/2}}, \quad (C7)$$

where we have taken into account the fact that  $F_\epsilon(z)$  and  $F_\epsilon^-(z)$  are real, and  $C$  is a contour along the real  $z$  axis. With the aid of Cauchy's theorem, the contour on the first integral can be moved up to  $C^+$  and that on the second integral down to  $C^-$  (see Fig. 9). By using Eqs. (C4) and (C6), the first integral can be written as

$$\int_{C^+} dz \frac{F_\epsilon(z)F_\epsilon^+(z)}{(\rho_i^2 + z^2)^{3/2}} = \int_{C^+} dz \frac{AA'}{\sqrt{k'(z)k(z)}} \times \exp\left(i \int_a^z k(z)dz - i \int_{a'}^z k'(z)dz\right), \quad (C8)$$

and, by using Eqs. (C5) and (C6), the second integral can be rewritten as

$$\int_{C^-} dz \frac{F_\epsilon(z)F_\epsilon^-(z)}{(\rho_i^2 + z^2)^{3/2}} = \int_{C^-} dz \frac{AA'}{\sqrt{k(z)k'(z)}} \times \exp\left(i \int_a^z k(z)dz - i \int_{a'}^z k'(z)dz + i\pi\right). \quad (C9)$$

For the case  $\epsilon > \epsilon'$ , the convention chosen in Sec. III, the integrands in Eqs. (C8) and (C9) are both exponentially small along their respective contours. Where the contours  $C^+$  and  $C^-$  run parallel to one another, the two integrands are equal and opposite; so the sum of the two integrals can be written as

$$\int_C dz \frac{F_\epsilon(z)F_\epsilon(z)}{(\rho_i^2 + z^2)^{3/2}} = \int_C dz \frac{AA'}{\sqrt{k(z)k'(z)}} \times \exp\left(i \int_a^z k(z)dz - i \int_{a'}^z k'(z)dz\right), \quad (C10)$$

where  $\tilde{C}$  is the loop contour shown in Fig. 10. The factor  $e^{+i\pi}$  in the integral (C9) was used to change the sign of  $dz$  along the lower half of this loop.

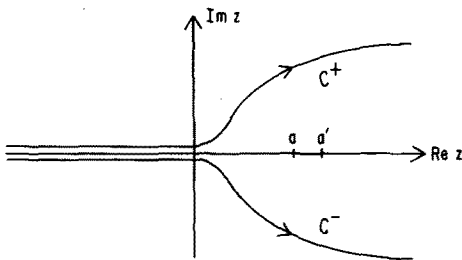


FIG. 9. Two contours,  $C^+$  and  $C^-$ , each a continuation of an integration along the real  $z$  axis.

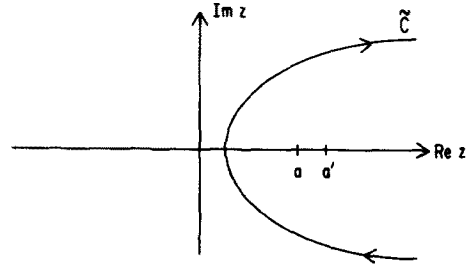


FIG. 10. The combined integration on  $C^+$  and  $C^-$  merged into a single contour  $\tilde{C}$  that describes a path of a reflected particle, analytically continued away from the classical turning points.

As mentioned earlier, the true solutions for  $F_\epsilon^+(z)$  and  $F_\epsilon^-(z)$  do not cancel exactly along the negative real axis, but add to the exponentially small quantity  $F_\epsilon(z)$ . Thus, neglecting the portion of the integral where  $C^+$  and  $C^-$  run parallel to one another involves an error of order  $|F_\epsilon^+(z)F_\epsilon^-(z)| \lesssim \exp[-(\epsilon/\hbar\Omega)(b\Omega/v_\parallel)]$ . We will see that this error is much smaller than the contribution to the integral along the contour  $\tilde{C}$ .

The integral along  $\tilde{C}$  can be rewritten as a time integral over the classical orbit. After substituting the relation  $k(z) - k'(z) = 2\Omega/[v_\parallel(z) + v_\parallel'(z)]$ , the integral is evaluated to lowest order in  $|k - k'|/k \sim \hbar\Omega/\epsilon$  simply by neglecting the distinction between primed and unprimed quantities. By introducing a time variable  $t$  through the relation  $dz/dt = v_\parallel(z)$ , Eq. (C10) reduces to

$$\int_C dz \frac{F_\epsilon(z)F_\epsilon(z)}{(\rho_i^2 + z^2)^{3/2}} \simeq \frac{A^2}{k} v_\parallel \int_{C_t} dt \frac{e^{i\Omega t}}{[\rho_i^2 + z^2(t)]^{3/2}}, \quad (C11)$$

where  $\tilde{C}_t$  is related to  $\tilde{C}$  through the mapping  $z = z(t)$ . In other words, the time integral is carried out in the complex  $t$  plane along the analytically continued path for a reflecting particle. This is the same integral that enters the classical analysis for the change in the adiabatic invariant, and, in Appendix B, the integral is shown to be of order  $\exp(-\pi\Omega b/2v_\parallel)$ .

To make  $A$  nonzero (and  $dn/d\epsilon$  and  $J_\epsilon^{-1}$  finite), we impose a potential barrier at  $z = L$ , where  $L$  is large but finite (i.e.,  $L \gg a$ ). The normalization condition for  $F_\epsilon(z)$  reduces to  $1 = \int_a^L dz |F_\epsilon(z)|^2 = 2A^2 L/k$ , and a discrete spectrum of energy values is specified by  $kL = \pi n$ . The density of states is given by  $dn/d\epsilon = (L/\pi)(dk/d\epsilon) = (1/2\pi\hbar)(2L/v_\parallel)$ , and the incident flux is given by  $J_\epsilon = v_\parallel/2L$ . For convenience we have imposed different boundary conditions here than were used in Sec. III, but  $dn/d\epsilon$  and  $J_\epsilon$  take the same values here as they do in Sec. III. Consequently, Eq. (39) contains the correct factors to give the transition probability for both choices of boundary conditions. In Eq. (39), the  $z$  integral appears with the factor  $2L/\sqrt{v_\parallel v_\parallel'}$ , and from Eq. (C11) and the condition  $A^2 = k/2L$  we find the result

$$\frac{2L}{\sqrt{v_\parallel v_\parallel'}} \int_C dz \frac{F_\epsilon(z)F_\epsilon(z)}{(\rho_i^2 + z^2)^{3/2}} = \int_{C_t} dt \frac{e^{i\Omega t}}{[\rho_i^2 + x^2(t)]^{3/2}}. \quad (C12)$$

This relation is the required generalization of Eq. (42).

In Sec. III, we considered collisions characterized by large parallel energy (i.e.,  $\epsilon, \epsilon' \gg T_{\parallel} \gg \hbar\Omega$ ) and large impact parameter (i.e.,  $\rho_i \gg b, b'$ ), and here we have considered collisions characterized by large parallel energy and small impact parameter (i.e.,  $\rho_i \ll b, b'$ ). In both cases, the matrix element is proportional to the time integral that enters the classical expression for the change in the adiabatic invariant. For the intermediate impact parameter (i.e.,  $\rho_i \sim b, b'$ ), the analysis is complicated by the fact that the positive and negative turning points (i.e.,  $\pm a$ ) must both be taken into account. Nevertheless, one can show that the matrix element is of order  $\exp[-(\pi/\sqrt{2})b\Omega/v_{\parallel}]$ , which is the order of the time integral for the case  $\rho_i = b$ . Thus, for any value of the impact parameter, the matrix element is proportional to the time integral as indicated in Eq. (42).

In Sec. II, the time integral was expressed as  $h(\kappa, \eta)\exp[-\kappa g(\eta)]$  where  $\kappa = \Omega b/v_{\parallel}$  and  $\eta = \rho/b$ . The plot of  $g(\eta)$  given in Fig. 1 shows that  $g(\eta)$  is a monotonic increasing function of  $\eta$ ; so the matrix element takes its largest value for the small impact parameter case (i.e.,  $\rho/b = \eta \ll 1$ ) that

was considered here. The case of small energy (i.e.,  $\epsilon, \epsilon' \sim \hbar\Omega \ll T_{\parallel}$ ) has not been discussed explicitly, since this case corresponds to large distance of closest approach (i.e.,  $b \sim e^2/\hbar\Omega \gg e^2/T_{\parallel}$ ) and very small matrix element.

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<sup>3</sup>L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1958), 1st ed., § 41, p. 144.

<sup>4</sup>J. H. Malmberg and T. M. O'Neil, *Phys. Rev. Lett.* **39**, 1333 (1977).

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<sup>6</sup>Reference 3, p. 67.

<sup>7</sup>G. R. Smith and A. N. Kaufmann, *Phys. Fluids* **21**, 2230 (1978).

<sup>8</sup>H. Goldstein, *Classical Mechanics* (Addison-Wesley, New York, 1980), 2nd ed., p. 383.

<sup>9</sup>Reference 3, p. 178.