Transport to thermal equilibrium of a pure electron plasma

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The transport of a pure electron plasma column across an axial magnetic field is considered, the transport being due to binary electron–electron collisions. The Boltzmann equation is expanded in inverse powers of the magnetic field, with the radial electric field and density gradient ordered self-consistently. The first nonvanishing terms in the particle flux involve the third derivative of the density and the second derivative of the electric field. An H theorem is used to show that this flux drives the plasma to a confined thermal equilibrium density profile. The electrons can come to thermal equilibrium with each other and still be confined magnetically, since the total canonical angular momentum is conserved. The final equilibrium can be predicted from the initial total number of particles, angular momentum, and energy. The approach to equilibrium is shown in a numerical example.

I. INTRODUCTION

This paper considers transport of a nonneutral electron plasma across a magnetic field, the transport being caused by electron-electron collisions. The plasma is assumed to be quiescent, cylindrically symmetric, and uniform in the axial direction. The axial magnetic field is assumed to be uniform and sufficiently large so that the Boltzmann equation may be expanded in inverse powers of the field.

An interesting property of the transport is that it leads to a state in which the electrons are in thermal equilibrium with each other, and yet are confined by the magnetic field. To see this, we note that electron-electron collisions conserve the total canonical angular momentum for the plasma. Consequently, the canonical angular momentum enters the expression for the thermal equilibrium distribution function on an equal footing with the energy.[1,2] The distribution is given by

$$f(r,\mathbf{V},t)=n_0\left(\frac{m}{2\pi T}\right)^{3/2}\times\exp\left\{-\frac{1}{T}\left[\frac{m\mathbf{V}^2}{2}-e\varphi(r)-\omega\left(mV_\theta r-\frac{e}{c}A_\theta r\right)\right]\right\}, \tag{1}$$

where n_0, ω, and T are parameters determined by the number of electrons, canonical angular momentum, and energy in the system. The electric potential $\varphi(r)$ may be related to the electron density through Poisson's equation, and the vector potential A_θ is given by $A_\theta = Br/2$. (The diamagnetic field is negligible, assuming all electron velocities are much less than c, and the density is below the Brillouin limit.) Here, m, $-e$, and $\mathbf{V}$ are the electron mass, charge, and velocity, and c is the speed of light.

The plasma is confined radially due to the vector potential term in the exponential (i.e., $\omega eA_\theta r/c=\omega eBr^2/2c$); this term forces the distribution to zero as r increases. (Note that for a neutral plasma, this thermal distribution would correspond to confinement for only one of the two oppositely charged species.) In a real confinement device, the walls of the cylindrical vacuum vessel would be outside the radius at which the distribution becomes exponentially small. The axial length of the plasma would be determined by end plates biased negatively relative to the cylindrical walls, so that the term $e\varphi$ in the exponential would make the distribution exponentially small near the end plates. Finite length experiments have been performed,[3] and the theory of finite length equilibria is described in the following paper.[4]

Another interesting property of the transport in a pure electron plasma is that the diffusive particle flux is not proportional to the density gradient. The flux is obtained by an expansion in inverse powers of the magnetic field, or equivalently in powers of the Larmor radius. The first term in the expansion, which is proportional to the first derivative of the density with respect to radius, vanishes identically. This occurs because conservation of momentum requires the guiding centers of two like particles to make equal and opposite steps in a collision.

The first nonvanishing term in the particle flux is a rather complicated expression involving the third derivative of the density. Similar results have been obtained in slab geometry by Simon,[5] Longmire and Rosenbluth,[6] and Braginskii[7]; these works differed slightly in numerical transport coefficients, and we resolve this discrepancy. We also find that the expression for the mobility flux is not simply proportional to the electric field, but rather involves the second derivative of the electric field. We shall see that these higher order expressions for the diffusion and mobility fluxes are just such that the electron distribution evolves to a thermal distribution of the form given by Eq. (1).

The heat flux is proportional to the temperature gradient, since the lowest order term in the expansion for the heat flux does not vanish. Since the heat flux enters in lower order than the particle flux, equal gradient scales for the temperature and density would produce a much larger heat flux than particle flux. Consequently, the plasma quickly becomes nearly isothermal, and remains so throughout the particle transport process. Of course, the temperature of the whole plasma may change

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as the density profile changes, since the electric field may do work on the plasma.

All this follows from mechanics and statistical mechanics, without reference to Poisson's equation. However, for a pure electron plasma in cylindrical geometry, the radial electric field appearing in the mobility flux is produced by the electrons themselves. Consequently, we must solve the electron transport equation self-consistently with Poisson's equation.

In Sec. II, the transport equations are obtained by expanding the Boltzmann equation in inverse powers of the magnetic field. We assume that the spatial gradient scale is no smaller than the Debye length. The conditions for validity of the expansion may then be stated as $\omega_p/\Omega \ll 1$ (i.e., the density is well below the Brillouin limit[2]), and $\nu_{ee}/\Omega \ll 1$, where $\omega_p \equiv (4\pi e^2 n_0/m)^{1/2}$ is the plasma frequency, $\Omega \equiv eB/mc$ is the electron gyrofrequency, and ν_{ee} is the electron-electron collision frequency. In the collision operator, the Rutherford scattering cross section is cut off for impact parameters less than the distance of closest approach and larger than the electron gyroradius.[8]

In Sec. III, the transport equations are shown to conserve particle number, canonical angular momentum, and energy. Also, an H theorem is developed to show that the electron density evolves monotonically toward an equilibrium. In Sec. IV, the thermal equilibrium distribution function of Eq. (1) is derived from the condition of zero transport. Equations are developed to predict the equilibrium parameters (n_0, T, ω) from the initial particle number, canonical angular momentum, and energy. Numerical solutions are obtained for the coupled transport and Poisson's equations in Sec. V, illustrating the approach to equilibrium. In the Appendix, we consider the results of Simon[5] and of Longmire and Rosenbluth,[6] and resolve the numerical discrepancy between these two works.

The work presented here is a theoretical description of a model, the model being based on certain assumptions. The assumption that the plasma is cylindrical and quiescent is the point where the model may differ significantly from the experimental situation. Under certain circumstances, wave-induced transport may dominate the collisional transport considered here.[9-11] Nevertheless, it is worthwhile to calculate the collisional transport as a benchmark, and to develop the interesting and unusual physics involved.

II. DERIVATION OF THE TRANSPORT EQUATIONS

As spatial variables in the Boltzmann equation, we choose cylindrical coordinates (r, θ, z). The velocity v used in this section is relative to the electron drift frame, and may be introduced in two steps. First, we define

$$v_z \equiv \hat{z}\cdot \mathbf{V}\ ,\quad v_r \equiv \hat{r}\cdot \mathbf{V}\ ,\quad v_\theta \equiv \hat{\theta}\cdot \mathbf{V} - v_d(r,t)\ ,$$

where $\mathbf{V}$ is the velocity in the laboratory frame and v_d is the drift velocity to be specified. The variables $v_\perp^2 \equiv v_r^2 + v_\theta^2$, $v^2 = v_\perp^2 + v_z^2$, and $v = |\mathbf{v}|$ are thus also relative to the drift frame.

In terms of these variables, the Boltzmann equation takes the form

$$\frac{\partial f}{\partial t} - \frac{\partial v_d}{\partial t}\frac{\partial f}{\partial v_\theta} + v_r\frac{\partial f}{\partial r} - r v_r \frac{\partial}{\partial r}\left(\frac{v_d}{r}\right)\frac{\partial f}{\partial v_\theta} + \left(\frac{v_d^2}{r} - \frac{eE}{m} - \Omega v_d\right)\frac{\partial f}{\partial v_r} - \left(\Omega - \frac{2v_d}{r} - \frac{v_\theta}{r}\right)\left(v_\theta\frac{\partial f}{\partial v_r} - v_r\frac{\partial f}{\partial v_\theta}\right) = C(f,f)\ .$$

Here, $E(r,t)$ is the radial electric field, $C(f,f)$ is the Boltzmann collision integral, and we have assumed axial and azimuthal symmetry (i.e., $\partial/\partial z = 0$, $\partial/\partial\theta = 0$). We choose the drift velocity $v_d(r,t)$ to be such that

$$v_d^2/r - eE/m - \Omega v_d = 0 \tag{2}$$

at each value of r. In other words, we work in a local drift frame in which the electric field is transformed out.

The second step in transforming velocity variables is to rewrite v_r and v_θ as

$$v_r = v_\perp \cos\beta\ ,\quad v_\theta = v_\perp \sin\beta\ .$$

The Boltzmann equation for $f(r, v_z, v_\perp, \beta, t)$ then takes the form

$$\frac{\partial f}{\partial t} - \frac{\partial v_d}{\partial t}\left(\sin\beta\frac{\partial}{\partial v_\perp} + \frac{\cos\beta}{v_\perp}\frac{\partial}{\partial\beta}\right)f + v_\perp\cos\beta\frac{\partial f}{\partial r} - r v_\perp \cos\beta\frac{\partial}{\partial r}\left(\frac{v_d}{r}\right)\left(\sin\beta\frac{\partial}{\partial v_\perp} + \frac{\cos\beta}{v_\perp}\frac{\partial}{\partial\beta}\right)f + \left(\Omega - \frac{2v_d}{r} - \frac{v_\perp}{r}\sin\beta\right)\frac{\partial f}{\partial\beta} = C(f,f)\ . \tag{3}$$

Before proceeding with the analysis of this equation, it is useful to specify the collision integral more completely. In our notation, the rate of change of a distribution $f(\mathbf{v})$ due to scattering from particles characterized by the distribution $g(\mathbf{v}_s)$ is given by

$$C(f,g) = \int d^3\mathbf{v}_s \int d\Omega_s\, \sigma(\Omega_s, u)\, u[f(\mathbf{v}')g(\mathbf{v}_s') - f(\mathbf{v})g(\mathbf{v}_s)]\ . \tag{4}$$

Here, $\sigma(\Omega_s, u)$ is the cross section for scattering angle Ω_s, $u \equiv |\mathbf{v} - \mathbf{v}_s|$, and the prime denotes post-collision velocities. Also, we will have occasion to use the notation $\tilde{C}(f,g) = C(f,g) + C(g,f)$.

Returning to Eq. (3), we note that each term has the dimensions of a frequency times the distribution function. We assume that Ω is the largest of the frequencies, and order the others relative to it. There are two independent small parameters in the ordering scheme. The first is associated with the electron-electron collision frequency: we define $C(f,f)/\Omega f = \delta$, where $\delta \ll 1$.

The second small parameter enters naturally through spatial scales. In considering the term $v_\perp\cos\beta(\partial f/\partial r)$, we anticipate that the r derivative will ultimately be expressed in terms of derivatives of the density and temperature. We assume that the density gradient scale is no smaller than the Debye length λ_D (see Sec. IV), so that $(v_\perp/\Omega)(1/n)(\partial n/\partial r) \approx r_L/\lambda_D = \omega_p/\Omega$, where the Larmor radius $r_L \equiv \bar{v}/\Omega \equiv (T/m\Omega^2)^{1/2}$. We denote $\omega_p/\Omega \equiv \epsilon$, and assume $\epsilon \ll 1$. As mentioned in the introduction, the temperature gradient is much weaker than the density gradient: we will see from the analysis that $(v_\perp/\Omega)(1/T)(\partial T/\partial r)=O(\epsilon^3)$. The terms involving v_d may also be expressed in terms of ϵ. Using the estimate $E\approx 4\pi enr$, the definition of v_d, and the assumption $\epsilon\ll 1$, we see that $v_d/\Omega r$ and $(r/\Omega)(\partial/\partial r)(v_d/r)$ are both $O(\epsilon^2)$. From the analysis, we will see that the time derivatives enter as $(1/\Omega)(\partial/\partial t)=O(\epsilon^4\delta)$. The ratio $v_d/\bar{v}$ is of order $\epsilon r/\lambda_D$, which we assume is no larger than unity.

In summary, we order the various frequencies as follows:

$$\frac{C(f,f)}{\Omega f}=O(\delta)\ ;\ \frac{v_\perp}{\Omega}\frac{1}{n}\frac{\partial n}{\partial r}=O(\epsilon)\ ,\ \frac{v_\perp}{\Omega r}\frac{\partial}{\partial\beta}=O(\epsilon)\ ;$$
$$\frac{v_d}{\Omega r}=O(\epsilon^2)\ ,\ \frac{r}{\Omega}\frac{\partial}{\partial r}\left(\frac{v_d}{r}\right)=O(\epsilon^2)\ ;\ \frac{v_\perp}{\Omega}\frac{1}{T}\frac{\partial T}{\partial r}=O(\epsilon^3)\ ; \tag{5}$$
$$\frac{1}{\Omega}\frac{1}{n}\frac{\partial n}{\partial t}=O(\epsilon^4\delta)\ ,\ \frac{1}{\Omega}\frac{1}{T}\frac{\partial T}{\partial t}=O(\epsilon^4\delta)\ ,\ \frac{1}{\Omega}\frac{1}{v_\perp}\frac{\partial v_d}{\partial t}=O(\epsilon^4\delta)\ .$$

We solve Eq. (3) by a perturbation expansion in powers of ϵ and δ. The expansion $f=\sum f_{n,m}$, where $f_{n,m}=O(\epsilon^n\delta^m)$, is substituted in Eq. (3), and each order is separately equated to zero. In lowest order, i.e., $O(\epsilon^0\delta^0)$, the equation is

$$\partial f_{00}/\partial\beta=0\ ,$$

which has the solution $f_{00}=f_{00}(r,v_z,v_\perp,t)$. To further constrain the function f_{00}, we go to the equation for order $\epsilon^0\delta^1$:

$$\partial f_{01}/\partial\beta=(1/\Omega)C(f_{00},f_{00})\ . \tag{6}$$

Since f_{01} is a periodic function of β, a constraint on f_{00} is

$$0=\int_0^{2\pi}d\beta\frac{\partial f_{01}}{\partial\beta}=\frac{2\pi}{\Omega}C(f_{00},f_{00})\ . \tag{7}$$

This, together with the previous conclusion that $f_{00}=f_{00}(r,v_z,v_\perp,t)$ implies that[12]

$$f_{00}=n(r,t)\left(\frac{m}{2\pi T(r,t)}\right)^{3/2}\exp\left(\frac{-mv^2}{2T(r,t)}\right)\ . \tag{8}$$

Here, the possibility of a drift in the z direction has been ignored, because of axial symmetry.

Constraints of the form in Eq. (7) are characteristic of the present type of perturbation expansion,[12] and they arise in each order. The transport equations will result from the constraint in order $\epsilon^4\delta^1$.

Since the right-hand side of Eq. (6) has been determined to be zero, the solution is $f_{01}=f_{01}(r,v_z,v_\perp,t)$. To further constrain the function, we go to the equation in order $\epsilon^0\delta^2$:

$$\partial f_{02}/\partial\beta=(1/\Omega)\tilde{C}(f_{00},f_{01})\ . \tag{9}$$

The constraint here is $\tilde{C}(f_{00},f_{01})=0$. Suppress for the moment the r and t dependence, and define $f_{01}(v_z,v_\perp)=f_{00}(v)P(v_z,v_\perp)$. Then using $f_{00}(v)f_{00}(v_s)=f_{00}(v')f_{00}(v_s')$, the constraint takes the form

$$\int d^3\mathbf{v}_s f_{00}(v)f_{00}(v_s)\int d\Omega_s\,\sigma u[P(v_z',v_\perp')+P(v_{sz}',v_{s\perp}')-P(v_z,v_\perp)-P(v_{sz},v_{s\perp})]=0\ .$$

In other words, $P(v_z,v_\perp)+P(v_{sz},v_{s\perp})$ must be conserved during the collision; so P must be of the form $P=A+Bv_z+Cv^2$.[12] Since A, B, and C are all small [i.e., $O(\delta)$], the addition of f_{01} to f_{00} corresponds only to a redefinition of the parameters n and T, and the possibility of a v_z drift which is ignorable because of axial symmetry. Consequently, we may set $f_{01}=0$. By extending the argument to higher order, one may set $f_{0n}=0$, for $n\geq 1$.

In order $\epsilon^1\delta^0$, Eq. (3) is

$$\frac{v_\perp}{\Omega}\cos\beta\frac{1}{n}\frac{\partial n}{\partial r}f_{00}+\frac{\partial f_{10}}{\partial\beta}=0\ .$$

Here, the constraint is satisfied automatically, since $\int_0^{2\pi}d\beta\cos\beta=0$. The solution for f_{10} is

$$f_{10}=h_{10}(r,v_z,v_\perp,t)-\frac{v_\perp}{\Omega}\sin\beta\frac{1}{n}\frac{\partial n}{\partial r}f_{00}\ , \tag{10}$$

where the function h contains all terms which are independent of β. To determine h_{10}, we go to the equation in order $\epsilon^1\delta^1$,

$$\frac{\partial f_{11}}{\partial\beta}=\frac{1}{\Omega}\tilde{C}(f_{00},h_{10})-\frac{1}{\Omega^2 n}\frac{\partial n}{\partial r}\tilde{C}(f_{00},v_\perp\sin\beta f_{00})\ . \tag{11}$$

The second term vanishes because of conservation of momentum. To see this, we simply write out the collision integral

$$\int d^3\mathbf{v}_s\int d\Omega_s\,\sigma u f_{00}(v)f_{00}(v_s)(v_\theta'+v_{s\theta}'-v_\theta-v_{s\theta})=0\ . \tag{12}$$

Here, we have recalled that $v_\perp\sin\beta=v_\theta$. Using the same arguments as those following Eq. (9), we may further conclude that $h_{10}=0$ and all $f_{1n}=0$, for $n\geq 1$.

In order $\epsilon^2\delta^0$, the equation is

$$\frac{K(r,t)}{2\Omega^2}v_\perp^2\sin 2\beta f_{00}+\frac{\partial f_{20}}{\partial\beta}=0\ ,$$

where

$$K(r,t)\equiv-\frac{1}{n}\frac{\partial^2 n}{\partial r^2}+\frac{1}{rn}\frac{\partial n}{\partial r}+\frac{m\Omega}{T}r\frac{\partial}{\partial r}\left(\frac{v_d}{r}\right)\ . \tag{13}$$

The constraint is satisfied automatically, and the solution is

$$f_{20}=h_{20}(r,v_z,v_\perp,t)+\frac{K}{4\Omega^2}v_\perp^2\cos 2\beta f_{00}\ . \tag{14}$$

In order $\epsilon^2\delta^1$, the equation is

$$\frac{\partial f_{21}}{\partial\beta}=\frac{1}{\Omega}\tilde{C}(f_{00},h_{20})+\frac{K}{4\Omega^3}\tilde{C}(f_{00},v_\perp^2\cos 2\beta f_{00})+\frac{1}{\Omega^3}\left(\frac{1}{n}\frac{\partial n}{\partial r}\right)^2 C(v_\perp\sin\beta f_{00},v_\perp\sin\beta f_{00})\ . \tag{15}$$

The last collision integral can be simplified by manipulating the integrand. One obtains

$$v_\theta'v_{s\theta}'-v_\theta v_{s\theta}=\tfrac{1}{2}(v_{s\theta}^2+v_\theta^2-v_{s\theta}'^2-v_\theta'^2)$$
$$=\tfrac{1}{4}(v_{s\perp}'^2\cos 2\beta_s'+v_\perp'^2\cos 2\beta'-v_{s\perp}^2\cos 2\beta_s-v_\perp^2\cos 2\beta)-\tfrac{1}{4}(v_{s\perp}'^2+v_\perp'^2-v_{s\perp}^2-v_\perp^2)\ ,$$

where we have used conservation of momentum and the identity $v_\perp^2\cos 2\beta=v_r^2-v_\theta^2$. The collision integral is thus

$C(v_\perp \sin\beta f_{00}, v_\perp \sin\beta f_{00})$

$= \frac{1}{4}\tilde{C}(f_{00}, v_\perp^2 \cos 2\beta f_{00}) - \frac{1}{4}\tilde{C}(f_{00}, v_\perp^2 f_{00})$.

Using this result, Eq. (15) may be rewritten as

$$\frac{\partial f_{21}}{\partial \beta} = \frac{M(r,t)}{4\Omega^3}\tilde{C}(f_{00}, v_\perp^2\cos 2\beta f_{00}) + \frac{1}{\Omega}\tilde{C}(f_{00}, h_{20}) - \frac{1}{4\Omega^3}\left(\frac{1}{n}\frac{\partial n}{\partial r}\right)^2 \tilde{C}(f_{00}, v_\perp^2 f_{00}) , \tag{16}$$

where

$$M(r,t) \equiv -r\frac{\partial}{\partial r}\left(\frac{1}{rn}\frac{\partial n}{\partial r} - \frac{m\Omega}{T}\frac{v_d}{r}\right) \approx -r\frac{\partial}{\partial r}\left(\frac{1}{rn}\frac{\partial n}{\partial r} + \frac{eE}{Tr}\right), \tag{17}$$

with the last expression for M keeping only terms lowest order in ϵ.

To carry out the β integration in Eq. (16), we note that all of the β dependence on the right-hand side is in the first collision integral. When written out, this integral is

$$\int d^3\mathbf{v}_s \int d\Omega_s\, \sigma(\Omega_s, u) u f_{00}(v) f_{00}(v_s) \times (v_\perp'^2 \cos 2\beta' + v_{s\perp}'^2 \cos 2\beta_s' - v_\perp^2 \cos 2\beta - v_{s\perp}^2 \cos 2\beta_s) .$$

Here, β_s is a variable of integration through $d\mathbf{v}_s$, and β' and β_s' are variables of integration through $d\Omega_s$. If we set $\beta_s = \beta + \Delta\beta_s$, $\beta' = \beta + \Delta\beta'$, and $\beta_s' = \beta + \Delta\beta_s'$, the operator

$$\int d^3\mathbf{v}_s \int d\Omega_s\, \sigma(\Omega_s, u) u f_{00}(v) f_{00}(v_s)$$

must be independent of β; it must take the form it would have for $\beta = 0$, with the variables of integration β_s, β', β_s' replaced by $\Delta\beta_s$, $\Delta\beta'$, $\Delta\beta_s'$. The β dependence is thus limited to the four cosine terms: $\cos(2\beta + 2\Delta\beta')$, $\cos(2\beta + 2\Delta\beta_s')$, $\cos(2\beta)$, and $\cos(2\beta + 2\Delta\beta_s)$. Integration with respect to β will give terms such as $\frac{1}{2}\sin(2\beta + 2\Delta\beta') = \frac{1}{2}\sin(2\beta')$, etc. These terms vanish upon integrating β from 0 to 2π, so the constraint in this order is that the second and third terms of Eq. (16) cancel each other. This will be so if h_{20} is chosen to be

$$h_{20} = \frac{1}{4\Omega^2}\left(\frac{1}{n}\frac{\partial n}{\partial r}\right)^2 \left(v_\perp^2 - \frac{2T}{m}\right) f_{00} . \tag{18}$$

The term with coefficient $2T/m$ merely amounts to a slight perturbation of $n(r)$, and could be deleted; we include it for notational simplicity in the rigid rotor analysis of Sec. IV. With the constraint satisfied, the solution for f_{21} is given by

$$f_{21} = \frac{M}{8\Omega^3}\tilde{C}(f_{00}, v_\perp^2 \sin 2\beta f_{00}) . \tag{19}$$

The constraint in order $\epsilon^2\delta^2$ has allowed us to set $h_{21} = 0$.

In order $\epsilon^3\delta^2$, the Boltzmann equation is

$$\frac{v_\perp}{\Omega}\cos\beta\left(\frac{mv^2}{2T} - \frac{3}{2}\right)\frac{f_{00}}{T}\frac{\partial T}{\partial r} + \frac{v_\perp \cos\beta}{4\Omega^3}\left(v_\perp^2 - \frac{2T}{m}\right)\frac{f_{00}}{n}\frac{\partial}{\partial r}\left[\frac{1}{n}\left(\frac{\partial n}{\partial r}\right)^2\right] + \frac{v_\perp^3}{4\Omega^3}\frac{f_{00}}{n}\left(\cos\beta\cos 2\beta\frac{\partial}{\partial r} + \sin\beta\sin 2\beta\frac{2}{r}\right)(nK) + \frac{2v_d}{\Omega^2 r} v_\perp \cos\beta\frac{f_{00}}{n}\frac{\partial n}{\partial r} + \frac{r}{\Omega^2} v_\perp \cos\beta\frac{\partial}{\partial r}\left(\frac{v_d}{r}\right)\left(1 - \frac{m}{T}v_\perp^2\sin^2\beta\right)\frac{f_{00}}{n}\frac{\partial n}{\partial r} + \frac{\partial f_{30}}{\partial \beta} = 0 . \tag{20}$$

The solution to Eq. (20) may be written as

$$\begin{aligned} f_{30} = h_{30}(r, v_z, v_\perp, t) &- \frac{v_\perp}{\Omega}\sin\beta f_{00}\left\{\frac{mv^2}{2T^2}\frac{\partial T}{\partial r} + \frac{v_\perp^2}{4\Omega^2 n}\frac{\partial}{\partial r}\left[\frac{1}{n}\left(\frac{\partial n}{\partial r}\right)^2\right] + \frac{v_\perp^2}{8\Omega^2 n}\frac{1}{r^2}\frac{\partial}{\partial r}r^2 nK - \frac{v_\perp^2}{4\Omega}\frac{m}{T}r\frac{\partial}{\partial r}\left(\frac{v_d}{r}\right)\frac{1}{n}\frac{\partial n}{\partial r}\right\} \\ &+ \frac{v_\perp}{\Omega}\sin\beta f_{00}\left\{\frac{3}{2T}\frac{\partial T}{\partial r} + \frac{T}{2\Omega^2 mn}\frac{\partial}{\partial r}\left[\frac{1}{n}\left(\frac{\partial n}{\partial r}\right)^2\right] - \frac{2v_d}{\Omega rn}\frac{\partial n}{\partial r} - \frac{r}{\Omega n}\frac{\partial}{\partial r}\left(\frac{v_d}{r}\right)\frac{\partial n}{\partial r}\right\} \\ &- \frac{v_\perp^3}{\Omega n}\sin 3\beta f_{00}\left[\frac{1}{24\Omega^2}\left(\frac{\partial}{\partial r} - \frac{2}{r}\right)(nK) + \frac{rm}{12\Omega T}\frac{\partial}{\partial r}\left(\frac{v_d}{r}\right)\frac{\partial n}{\partial r}\right]. \end{aligned} \tag{21}$$

Only the first term after h_{30} will actually contribute to the transport.

In order $\epsilon^3\delta^1$, the equation is

$$\frac{v_\perp}{\Omega}\cos\beta\frac{\partial f_{21}}{\partial r} - \frac{v_\perp}{\Omega r}\sin\beta\frac{\partial f_{21}}{\partial \beta} + \frac{\partial f_{31}}{\partial \beta} = \frac{1}{\Omega}\tilde{C}(f_{00}, f_{30}) + \frac{1}{\Omega}\tilde{C}(f_{10}, f_{20}) . \tag{22}$$

We do not need to solve this equation for f_{31}; we will only need two moments of it. Multiplying by $v_\perp \sin\beta$ and integrating over velocity gives

$$\int d^3\mathbf{v}\, v_r f_{31} = \frac{1}{2\Omega}\frac{1}{r^2}\frac{\partial}{\partial r}r^2 \int d^3\mathbf{v}\, v_\perp^2 \sin 2\beta f_{21} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2 \frac{M(r,t)}{16\Omega^4}\int d^3\mathbf{v}\, 2v_r v_\theta \tilde{C}(f_{00}, 2v_r v_\theta f_{00}) . \tag{23}$$

To obtain this, we have integrated by parts in the β integral (i.e., $\int d^3\mathbf{v} = \int dv_z \int v_\perp dv_\perp \int d\beta$), used $\int d^3\mathbf{v}\, v_\perp \sin\beta \tilde{C}(f, g) = 0$, and substituted from Eq. (19) for f_{21}. For brevity, we write v_r, v_θ for $v_\perp\cos\beta$, $v_\perp\sin\beta$. One can recognize the left-hand side of Eq. (23) as the radial flux of particles.

Multiplying Eq. (22) by $\frac{1}{2}mv^2 v_\perp \sin\beta$ and integrating over velocity gives

$$\int d^3\mathbf{v}\, \tfrac{1}{2}mv^2 v_r f_{31} = \frac{1}{2\Omega}\frac{1}{r^2}\frac{\partial}{\partial r}r^2 \int d^3\mathbf{v}\, \tfrac{1}{2}mv^2 v_\perp^2 \sin 2\beta f_{21} - \frac{1}{\Omega}\int d^3\mathbf{v}\, \tfrac{1}{2}mv^2 v_\perp \sin\beta[\tilde{C}(f_{00}, f_{30}) + \tilde{C}(f_{10}, f_{20})]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{M}{16\Omega^4}\int d^3\mathbf{v}\,\tfrac{1}{2}mv^2 2v_r v_\theta \tilde{C}(f_{00}, 2v_r v_\theta f_{00}) + \frac{1}{\Omega^2 T}\frac{\partial T}{\partial r}\int d^3\mathbf{v}\,\tfrac{1}{2}mv^2 v_\theta \tilde{C}\Big(f_{00}, \frac{mv^2}{2T}v_\theta f_{00}\Big)$$
$$+ \frac{1}{4\Omega^4}\left\{\frac{1}{n}\frac{\partial}{\partial r}\left[\frac{1}{n}\left(\frac{\partial n}{\partial r}\right)^2\right] + \frac{1}{2n}\frac{1}{r^2}\frac{\partial}{\partial r} r^2 nK - \frac{m\Omega}{T} r \frac{\partial}{\partial r}\left(\frac{v_d}{r}\right)\frac{1}{n}\frac{\partial n}{\partial r}\right\}\int d^3\mathbf{v}\frac{mv^2}{2} v_\theta \tilde{C}(f_{00}, v_\perp^2 v_\theta f_{00})$$
$$+ \frac{1}{4\Omega^4}\left(\frac{1}{n}\frac{\partial n}{\partial r}\right)^3\int d^3\mathbf{v}\frac{mv^2}{2}v_\theta \tilde{C}(v_\theta f_{00}, v_\perp^2 f_{00}) + \frac{K}{4\Omega^4}\frac{1}{n}\frac{\partial n}{\partial r}\int d^3\mathbf{v}\frac{mv^2}{2}v_\theta \tilde{C}(v_\theta f_{00}, v_\perp^2 \cos 2\beta f_{00}) \ . \qquad (24)$$

To obtain this, we have used $\tilde{C}(f_{00}, v_\theta f_{00}) = 0$ and $\int_0^{2\pi} d\beta \sin\beta \tilde{C}(f_{00}, v_\perp^3 \sin 3\beta f_{00}) = 0$. One can recognize the left-hand side of Eq. (24) as the radial heat flux.

We do not need the equation for order $\epsilon^4\delta^0$. We need only realize that the average of f_{40} over β contains a term h_{40} which is to be adjusted according to the constraint on the equation in order $\epsilon^4\delta^1$. This constraint is given by

$$\frac{\partial f_{00}}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r} r\int_0^{2\pi}\frac{d\beta}{2\pi} v_r f_{31} - r\frac{\partial}{\partial r}\left(\frac{v_d}{r}\right)\left(1 + \frac{v_\perp}{2}\frac{\partial}{\partial v_\perp}\right)\int_0^{2\pi}\frac{d\beta}{2\pi}\sin 2\beta f_{21} = \int_0^{2\pi}\frac{d\beta}{2\pi}[\tilde{C}(f_{00}, f_{40}) + \tilde{C}(f_{10}, f_{30}) + C(f_{20}, f_{20})] \ , \qquad (25)$$

where the last two terms on the left-hand side have been obtained using integration by parts.

As in the Chapman–Enskog theory,[12] we imagine both sides of Eq. (25) to be expanded in terms of the eigenfunctions $\Psi_l(v_z, v_\perp)$ defined by $\tilde{C}(f_{00}, \Psi_l f_{00}) = \lambda_l(r)\Psi_l f_{00}$. Writing $h_{40} = f_{00}\sum a_l \Psi_l$, we see that the coefficients a_l may be adjusted so that Eq. (25) is satisfied, except for l corresponding to zero eigenvalue (i.e., $\lambda_l = 0$). There are three eigenfunctions which have zero eigenvalue and are also β independent: $\Psi = 1$, $\Psi = v^2$, and $\Psi = v_z$. It is to satisfy the components of Eq. (25) corresponding to these eigenfunctions that we must introduce the time dependence (i.e., $\partial f_{00}/\partial t$) in this order.

Projecting Eq. (25) on $\Psi = 1$, that is, integrating over $d^3\mathbf{v}$, yields the particle transport equation

$$\frac{\partial n}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r} r\int d^3\mathbf{v}\, v_r f_{31} = 0 \ . \qquad (26)$$

Here, we have integrated by parts to eliminate the third term in Eq. (25), and have used $\int d^3\mathbf{v}\,\tilde{C}(f, g) = 0$. Projecting Eq. (25) on $\Psi = \frac{1}{2}mv^2$ yields the heat transport equation,

$$\frac{\partial}{\partial t}\left(\frac{3}{2}nT\right) + \frac{1}{r}\frac{\partial}{\partial r} r\int d^3\mathbf{v}\frac{mv^2}{2} v_r f_{31} + r\frac{\partial}{\partial r}\left(\frac{v_d}{r}\right)\int d^3\mathbf{v}\, m v_r v_\theta f_{21} = 0 \ , \qquad (27)$$

where we used $\int d^3\mathbf{v}\,\frac{1}{2}mv^2\tilde{C}(f, g) = 0$. The projection on $\Psi = v_z$ is satisfied trivially, because of the assumption of axial symmetry.

Finally, we must evaluate the velocity integrals in the expressions for the radial particle and heat transport equations [i.e., Eqs. (26) and (27)]. The radial particle flux of Eq. (26) was given in Eq. (23). To evaluate the collision integral in Eq. (23), we note that

$$\tilde{C}(f_{00}, 2v_r v_\theta f_{00}) = \int d^3\mathbf{v}_s f_{00}(v) f_{00}(v_s) \times \int d\Omega_s\, \sigma u 2(v_r' v_\theta' + v_{sr}' v_{s\theta}' - v_r v_\theta - v_{sr} v_{s\theta})$$
$$= \int d^3\mathbf{v}_s f_{00}(v) f_{00}(v_s) 2\int d\Omega_s\, \sigma u \hat{r}\cdot(\mathbf{u}'\mathbf{u}' - \mathbf{u}\mathbf{u})\cdot\hat{\theta} \ .$$

Here, we have used the velocities $\mathbf{u} \equiv \mathbf{v} - \mathbf{v}_s$ and $\mathbf{w} \equiv \frac{1}{2}(\mathbf{v} + \mathbf{v}_s)$, and the identity $\mathbf{w}' = \mathbf{w}$. The scattering is described in terms of a scattering angle ψ and azimuthal angle φ, as $\mathbf{u}' = (\hat{u}\cos\psi + \hat{m}\sin\psi\sin\varphi + \hat{n}\sin\psi\cos\varphi)u'$, where $\hat{u}$, $\hat{m}$, $\hat{n}$ are mutually orthogonal unit vectors. Using the Rutherford formula for the cross section σ, the scattering integral is

$$\int d\Omega_s\, \sigma u 2\hat{r}\cdot(\mathbf{u}'\mathbf{u}' - \mathbf{u}\mathbf{u})\cdot\hat{\theta}$$
$$= \int_{\psi_1}^{\psi_2} 2\pi\sin\psi\, d\psi \frac{e^4}{m^2 u^4 \sin^4(\psi/2)} u\hat{r} \cdot\left[\hat{u}\hat{u}(\cos^2\psi - 1) + \left(\frac{\hat{m}\hat{m}}{2} + \frac{\hat{n}\hat{n}}{2}\right)\sin^2\psi\right]\cdot\hat{\theta}$$
$$= \frac{16\pi e^4}{m^2 u}\ln\left(\frac{\sin\psi_2}{\sin\psi_1}\right)\hat{r}\cdot(-2\hat{u}\hat{u} + \hat{m}\hat{m} + \hat{n}\hat{n})\cdot\hat{\theta}$$
$$\equiv \frac{16\pi e^4}{m^2 u^3}\ln\Lambda(-3u_r u_\theta) \ .$$

Here, we have used the relation $\hat{u}\hat{u} + \hat{m}\hat{m} + \hat{n}\hat{n} = \hat{z}\hat{z} + \hat{r}\hat{r} + \hat{\theta}\hat{\theta}$. The parameter Λ is the ratio of the maximum to minimum impact parameters; we take these to be the Larmor radius and the distance of closest approach, giving $\Lambda = \bar{v}T/e^2\Omega$.[8] The two velocity integrals in Eq. (23) may be simply evaluated in center-of-mass coordinates, as

$$\int d^3\mathbf{v}\, 2v_r v_\theta \tilde{C}(f_{00}, 2v_r v_\theta f_{00}) = \frac{16\pi e^4}{m^2}\ln\Lambda\left(\frac{T}{\pi m}\right)^{1/2}\left(-\frac{6}{15}n^2\right) \ . \qquad (28)$$

The particle transport equation may now be given explicitly. Substituting Eqs. (28), (23), and (17) into Eq. (26) gives

$$\frac{\partial n}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r} r\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \beta T^{1/2} r n^2 \frac{\partial}{\partial r}\frac{1}{r}\left(\frac{1}{n}\frac{\partial n}{\partial r} + \frac{eE}{T}\right)\right] = 0 \ , \qquad (29)$$

where

$$\beta T^{1/2} \equiv \frac{3}{8}\frac{\bar{v}^4}{\Omega^4}\left(\frac{16\sqrt{\pi}e^4}{15 m^2 \bar{v}^3}\ln\Lambda\right) \equiv \frac{3}{8}\frac{\bar{v}^4}{\Omega^4}\frac{\nu_{ee}}{n} \ . \qquad (30)$$

Except for the logarithmic dependence on $\Lambda(T)$, β is a constant determined by physical constants and the magnetic field. The term in the outer-most brackets of Eq. (29) is the radial particle flux, which may be conveniently written

$$J(r,t) = -\beta T^{1/2}\frac{1}{r^2}\frac{\partial}{\partial r}r^2 n^2(r,t)M(r,t) \tag{31}$$

using Eq. (17). We have ignored the small variation of temperature with radius.

Unlike the usual diffusion and mobility results,[13,14] the particle flux involves the third derivative of the density and the second derivative of the electric field. A diffusive flux proportional to $\partial n/\partial r$ would normally appear through f_{11}; however, conservation of momentum in like particle collisions implies that this term is zero, as shown in Eqs. (11) and (12). We see no mobility proportional to the electric field, as would occur with electron-neutral collisions, because the drift velocity does not result in momentum being lost from the electrons.

Expressions similar to Eq. (29) have been obtained for like-particle transport in slab geometry,[5-7] but different methods have given different numerical coefficients. We resolve this discrepancy in the Appendix.

The integrals in the heat transport equation may be evaluated similarly to those in the particle transport equation. Substituting Eqs. (24) and (19) into Eq. (27) and evaluating the integrals yields

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{3}{2}nT\right) - \frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{5}{2}\frac{\bar{v}^2}{\Omega^2}\nu_{ee}\, n\frac{\partial T}{\partial r}\right) &= \frac{1}{r}\frac{\partial}{\partial r}r\left(-\frac{49}{12}JT - \frac{14}{3}\beta T^{1/2}TMn\frac{\partial n}{\partial r}\right) \\ &\quad + \beta T^{1/2} r\frac{\partial}{\partial r}\left(\frac{v_d}{r}\right)\Omega m n^2 M \,.\end{aligned} \tag{32}$$

The heat flux term on the left-hand side would cause a thermal gradient with scale $(r_L/T)(\partial T/\partial r) \approx \epsilon$ to diffuse at a rate $(1/\Omega T)(\partial T/\partial t) \approx \epsilon^2\delta$. Thus, the plasma relaxes to a nearly isothermal state before particle transport, scaling as $(1/\Omega n)(\partial n/\partial t) \approx \epsilon^4\delta$, can occur. When particle transport does occur, the driving terms on the right-hand side will maintain a small thermal gradient, $(r_L/T)(\partial T/\partial r) \approx \epsilon^3$, as anticipated in Eq. (5).

III. CONSERVATION THEOREMS AND H THEOREM

In this section, we consider the general properties of the transport equations of Sec. II. We show that the transport equations conserve the number of particles, the canonical angular momentum, and the energy. Further, we derive an H theorem which demonstrates that the transport progresses monotonically toward an equilibrium. Since the transport equations give time derivatives only to lowest order in ϵ, this section will take only the lowest order terms of the quantities of interest.

First, the particle transport equation (29) is easily seen to conserve the number of particles (per unit length in the z direction):

$$\frac{\partial}{\partial t}N(t) = \frac{\partial}{\partial t}\int_0^\infty 2\pi r\,dr\,n(r,t) = -\int_0^\infty 2\pi r\,dr\frac{1}{r}\frac{\partial}{\partial r}rJ(r,t) = 0\,. \tag{33}$$

Here, we have integrated by parts, observing that the particle flux vanishes at $r=0$ and $r=\infty$.

The canonical angular momentum is given by

$$\begin{aligned}L(t) &= \int_0^\infty 2\pi r\,dr\int d^3\mathbf{V} f(r,\mathbf{V})\left(mV_\theta r - \frac{m}{2}\Omega r^2\right) \\ &\approx -\frac{m\Omega}{2}\int_0^\infty 2\pi r\,dr\,r^2 n(r,t)\,,\end{aligned} \tag{34}$$

since $\langle V_\theta\rangle$ is small compared with Ωr. The time derivative is then

$$\begin{aligned}\frac{\partial L}{\partial t} &= \frac{m\Omega}{2}\int_0^\infty 2\pi r\,dr\,r^2\frac{1}{r}\frac{\partial}{\partial r}rJ \\ &= \frac{m\Omega}{2}\beta T^{1/2}\int_0^\infty 2\pi r\,dr\frac{2}{r}\frac{\partial}{\partial r}r^2 n^2 M = 0\,,\end{aligned}$$

where we have integrated by parts and used Eq. (31).

Conservation of energy can be formulated by integrating Eq. (32) with respect to $r\,dr$. The terms beginning with the operator $(1/r)(\partial/\partial r)$ integrate to zero. Neglecting any spatial variation in T, we obtain

$$\begin{aligned}\frac{3}{2}N\frac{\partial T}{\partial t} &= \beta T^{1/2}\int_0^\infty 2\pi r\,dr\,r\frac{\partial}{\partial r}\left(\frac{v_d}{r}\right)\Omega m n^2 M \\ &= -\beta T^{1/2}\int_0^\infty 2\pi\,dr\frac{mv_d}{r}\frac{\partial}{\partial r}(r^2 n^2 M) \\ &= \int_0^\infty 2\pi r\,dr(-eE)J\,.\end{aligned} \tag{35}$$

Here, we have integrated by parts, approximated v_d by $-eE/m\Omega$, used $dN/dt = 0$ and Eq. (31). It is apparent that to lowest order in ϵ, the change in thermal energy equals the work done by the electric field. This will be true even if external charges contribute to the electric field, as would be the case if a potential were placed on a thin wire at $r=0$.

However, we are most interested in the particular case where the electric field is generated entirely by the charge density n. Then, we may use the time derivative of Poisson's equation,

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial E}{\partial t} = -4\pi e\frac{\partial n}{\partial t}\,,$$

together with the continuity equation, to obtain

$$\frac{3}{2}N\frac{\partial T}{\partial t} = \int_0^\infty 2\pi r\,dr(-eE)\frac{\partial}{\partial t}\left(\frac{E}{4\pi e}\right) = -\frac{\partial}{\partial t}\int_0^\infty 2\pi r\,dr\frac{E^2}{8\pi}\,. \tag{36}$$

The integral on the right-hand side is the total electrostatic energy per unit length of the column of charge; this energy is logarithmically infinite due to the contributions at large radii. We assume that the column is surrounded at a large radius R_w by a conducting cylinder carrying a charge per unit length of $+Ne$, and define the total energy to be

$$W \equiv \frac{3}{2}NT + \int_0^{R_w} 2\pi r\,dr\frac{E^2}{8\pi}\,. \tag{37}$$

This energy is finite, and $dW/dt = 0$. We assume that R_w is much larger than the radial extent of the plasma, so that $n(R_w, t)$ is essentially zero. The particle number and angular momentum integrals of Eqs. (33) and (34) are therefore the same whether one integrates to R_w or to ∞. We note that W does not include the kinetic energy associated with the drift velocity (i.e., $\int 2\pi r\, dr\, n\frac{1}{2}mv_d^2$); this energy is order ϵ^2 compared with the electrostatic energy, but may be comparable to the thermal energy if $\epsilon r/\lambda_D = O(1)$.

We can gain some insight into the evolution implied by the transport equations by considering the quantity H given by

$$H(t) = \int_0^\infty 2\pi r\, dr\, n \ln(nT^{-3/2}) \ . \tag{38}$$

This definition of H is motivated by the fact that the entropy of a uniform gas would be proportional to $-H$. Since R_w may be arbitrarily large, we take the limit of integration to be ∞ in the following discussion. The time derivative of H is then

$$\begin{aligned}\frac{dH}{dt} &= \int_0^\infty 2\pi r\, dr \frac{\partial n}{\partial t}\ln(n) - \frac{3}{2}\frac{N}{T}\frac{\partial T}{\partial t} \\ &= -\int_0^\infty 2\pi r\, dr \frac{1}{r}\frac{\partial}{\partial r}(rJ)\ln(n) + \frac{1}{T}\int_0^\infty 2\pi r\, dr\, eEJ \\ &= \int_0^\infty 2\pi r\, dr\, J\left(\frac{1}{n}\frac{\partial n}{\partial r} + \frac{eE}{T}\right),\end{aligned}$$

where we have used Eqs. (29) and (35), and integrated by parts. Again integrating by parts and using Eq. (17) for M, we obtain

$$\begin{aligned}\frac{dH}{dt} &= -\beta T^{1/2}\int_0^\infty 2\pi r\, dr \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 n^2 M)r\left(\frac{1}{rn}\frac{\partial n}{\partial r} + \frac{eE}{rT}\right) \\ &= \beta T^{1/2}\int_0^\infty 2\pi r\, dr (n^2 M)(-M) \le 0 \ . \end{aligned} \tag{39}$$

Thus, H can never increase; it either decreases or remains constant with time.

The function H cannot continue to decrease indefinitely, since it may be shown to remain finite. To establish this, we expand the logarithm in Eq. (38), and show that each of the two contributions to H are bounded below. First, suppose that $\int_0^\infty 2\pi r\, dr\, n\ln(n)$ is *not* bounded. Then, since $-\int_0^\infty 2\pi r\, dr\, nr^2 = 2L/m\Omega$ is finite, $\ln(n)$ must be more negative than $-r^2$ as $r\to\infty$; this implies $n \le \exp(-r^2)$, in which case $\int_0^\infty 2\pi r\, dr\, n\ln(n)$ certainly *is* bounded. The integral must therefore be bounded.

Second, we must show that $\int_0^\infty 2\pi r\, dr\, n\ln(T^{-3/2}) = -(3/2)N\ln(T)$ is bounded below, which will be the case if T is bounded above. T will be bounded if the possible change in the electrostatic energy integral of Eq. (37) is always finite, even though R_w may be arbitrarily large. The change in electrostatic energy $\delta\mathcal{E}_0$ within a finite radius R_0 is certainly finite, since all charge densities remain finite. For the contribution from large radii, we may write

$$\delta\int_0^\infty 2\pi r\, dr\frac{E^2}{8\pi} = \delta\mathcal{E}_0 + e^2\int_{R_0}^\infty \frac{dr}{r}[2N(r)\delta N(r) + \delta N^2(r)] \ ,$$

where $N(r)$ and $\delta N(r)$ are the number and change in number of electrons per unit length within a radius r. Since $\delta L = 0$, $\delta n(r)$ must approach zero faster than r^{-4}. Then $\delta N(r)$ approaches zero faster than r^{-2}, and the integrals of δN are seen to be finite. Thus, the function H is bounded below.

Since H cannot decrease indefinitely, it must necessarily approach an equilibrium with $dH/dt = 0$. From Eq. (39), it is apparent that $dH/dt = 0$ if, and only if, $M(r,t) = 0$. From the particle transport equations, (29) and (31), one can see that $M(r,t) = 0$ implies $\partial n/\partial t = 0$. The heat transport equation further implies that $\partial T/\partial r = \partial T/\partial t = 0$. (Here, one has in mind that the plasma evolves to an isothermal state on a heat conduction time scale, which is negligible compared with the particle transport time scale.) The H theorem thus implies that the plasma evolves monotonically toward an isothermal equilibrium with the density and temperature independent of time.

IV. EQUILIBRIUM STATES

We now characterize the equilibria determined by $M(r,t) = 0$ and $T(r,t) = \text{const}$. Using Eq. (17) for M, the density profile must vary as

$$\frac{1}{n}\frac{\partial n}{\partial r} - \frac{e}{T}\frac{\partial\varphi}{\partial r} = -2\alpha r \ , \tag{40}$$

where α is an arbitrary constant. This may be integrated to give

$$n(r) = n_0\exp[(e/T)\varphi(r) - \alpha r^2] \ , \tag{41}$$

with n_0 also arbitrary. An equilibrium density profile in the presence of an arbitrary electrostatic potential $\varphi(r)$ must necessarily be of this form. This density profile may also be obtained by noting that the equilibrium must be a Gibbs distribution.[2]

Further, the Boltzmann equation results of Sec. II enable us to directly calculate the equilibrium velocity distribution. For clarity, we work only to order ϵ^2. Using Eq. (40) for $\partial\varphi/\partial r$, the local drift velocity of Eq. (2) may be written

$$v_d = \omega r + \frac{T}{m\Omega n}\frac{\partial n}{\partial r} \ ,$$

where the angular frequency ω is determined by

$$\alpha = (m/2T)(\Omega\omega - \omega^2) \ . \tag{42}$$

Equations (8), (10), (14), (18), and (19) then give the velocity distribution

$$\begin{aligned} f(r,\mathbf{v}) &= f_{00} + f_{10} + f_{20} \\ &= f_{00}\left[1 + \frac{v_\theta}{\Omega n}\frac{\partial n}{\partial r} + \frac{1}{2\Omega^2 n^2}\left(\frac{\partial n}{\partial r}\right)^2\left(v_\theta^2 - \frac{T}{m}\right)\right] \\ &= f_{00}\exp\left[-\frac{v_\theta}{\Omega n}\frac{\partial n}{\partial r} - \frac{1}{2\Omega^2 n^2}\left(\frac{\partial n}{\partial r}\right)^2\frac{T}{m}\right] . \end{aligned}$$

Expressed in terms of the laboratory velocity $\mathbf{V} = \mathbf{v} + v_d\hat{\theta}$, this is

$$f(r,\mathbf{V}) = n(r)\left(\frac{m}{2\pi T}\right)^{3/2}\exp\left\{-\frac{m}{2T}[V_r^2 + V_z^2 + (V_\theta - \omega r)^2]\right\} . \tag{43}$$

The distribution is seen to be a Maxwellian rotating as a rigid rotor, that is, with average angular velocity $\langle V_\theta \rangle = \omega r$.

Using Eq. (41) for $n(r)$, the distribution function may be written as

$$f(r, \mathbf{V}) = n_0 \left(\frac{m}{2\pi T}\right)^{3/2} \times \exp\left\{-\frac{1}{T}\left[\frac{m}{2}\mathbf{V}^2 - e\varphi - \omega\left(mV_\theta r - \frac{m}{2}\Omega r^2\right)\right]\right\} ; \quad (44)$$

this is the thermal equilibrium distribution anticipated in Eq. (1). The argument of the exponential is seen to be $-(\mathcal{H} - \omega p_\theta)/T$, where $\mathcal{H}$ is the energy of a single particle, and $p_\theta = mV_\theta r - eA_\theta r/c$ is the canonical angular momentum of the particle. The equilibrium $f(r, \mathbf{V})$ is thus seen to be a Gibbs distribution. The energy and momentum enter the distribution on an equal basis, since both are conserved quantities.

The combination $\mathcal{H} - \omega p_\theta$ may be written

$$\mathcal{H} - \omega p_\theta = \frac{m}{2}[V_r^2 + V_z^2 + (V_\theta - \omega r)^2] - e\varphi + \frac{m}{2}\omega\Omega r^2 - \frac{m}{2}\omega^2 r^2 . \quad (45)$$

The first term on the right-hand side constitutes the kinetic energy in the rigid rotor frame. The remaining terms may be thought of as the electrostatic, magnetic, and centrifugal potentials. The magnetic potential arises from the $\omega r \times B$ magnetic force which appears as an induced electric field in the rotating frame. Similarly, the fictitious centrifugal force gives rise to a potential which is small (order ϵ^2) compared with the magnetic potential.

The equilibrium density profile of Eq. (41) is valid for arbitrary potential $\varphi(r)$, even if external charges contribute to it. For the particular case in which φ arises solely from the charge density n, we use Poisson's equation,

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\varphi(r) = 4\pi e n(r) , \quad (46)$$

with $\varphi(0) = \partial\varphi/\partial r|_{r=0} = 0$.

The two equations, (41) and (46), contain three parameters: n_0, T, and α. The temperature T merely sets the scale for the density and potentials, so we may write Eqs. (41) and (46) as

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\frac{e\varphi}{T} = \frac{4\pi e^2 n_0}{T}\exp\left(\frac{e\varphi}{T} - \alpha r^2\right) . \quad (47)$$

Further, n_0/T may be incorporated into the spatial scale by considering the radius measured in units of the central Debye length, as

$$\rho \equiv r/\lambda_D , \quad \lambda_D^2 \equiv T/4\pi e^2 n_0 . \quad (48)$$

We define

$$\psi(\rho) \equiv [e\varphi(\rho)/T] - \alpha\rho^2\lambda_D^2 , \quad (49)$$

giving $n(\rho) = n_0 \exp[\psi(\rho)]$. Then, Eq. (47) may be written

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\psi = e^\psi - 4\alpha\lambda_D^2 \equiv e^\psi - (1+\gamma) , \quad (50)$$

with

$$\gamma \equiv 4\alpha\lambda_D^2 - 1 = \frac{m(\Omega\omega - \omega^2)}{2\pi e^2 n_0} - 1 ,$$

$$\psi(0) = 0 , \quad \left.\frac{\partial\psi}{\partial\rho}\right|_{\rho=0} = 0 .$$

Equation (50) is completely solved by the family of solutions parameterized by the single variable γ. Thus, all possible thermal equilibrium profiles are described by $n(r/\lambda_D) = n_0 \exp[\psi(r/\lambda_D;\gamma)]$. Bounded solutions for which $n(r) \to 0$ as $r \to \infty$ exist only for $\gamma > 0$. Representative profiles of $\psi(\rho)$ and $n(\rho)$ are shown in Fig. 1, for $10^{-5} \le \gamma \le 1$; these curves were obtained by numerical integration. For $\gamma \ll 1$, the density is quite constant out to a certain radius, then falls toward zero in several Debye lengths; the parameter γ merely determines the radial extent of the plasma in Debye lengths.

We may interpret the density profile as follows. The magnetic and centrifugal potentials appearing in Eq. (45) may be thought of as arising from a hypothetical cylinder of uniform positive charge density

$$n_+ = m(\Omega\omega - \omega^2)/2\pi e^2 = (1+\gamma)n_0 .$$

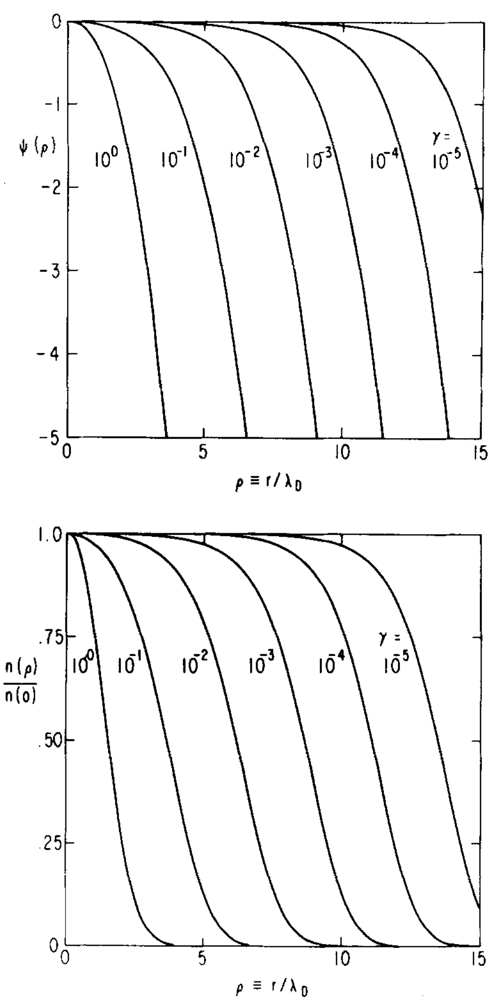

FIG. 1. The family of self-consistent thermal equilibrium density profiles $n(\rho) = n_0 \exp[\psi(\rho;\gamma)]$, for several values of the parameter γ. The radius is scaled as $\rho = r/\lambda_D$.

For $\gamma \ll 1$, the electron density almost exactly "neutralizes" the hypothetical positive charge out to some radius at which the supply of electrons is exhausted. At that radius the electron density falls off on a scale set by the Debye length.

Some analytic insight into these profiles may be obtained for $\gamma \ll 1$ and $\psi \ll 1$. Expanding $\exp(\psi) \approx 1 + \psi$, Eq. (50) becomes

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial\psi}{\partial\rho} = \psi - \gamma ,$$

with the same boundary conditions. This has the solution

$$\psi = \gamma[1 - I_0(\rho)] \approx \gamma[1 - (2\pi\rho)^{-1/2} e^{\rho}] ,$$

where I_0 is the modified Bessel function, and the second expression is valid for $\rho \gg 1$. The density is thus constant in the region where ψ grows exponentially from a (negative) small value. The density has fallen to 1/2 its central value when $\psi = -\ln(2)$, this occurring at a radius R_p given approximately by

$$\ln(2)(2\pi R_p/\lambda_D)^{1/2} \exp(-R_p/\lambda_D) = \gamma . \quad (51)$$

The physical quantities N, L, and W may be parametrized by the single variable γ, with the temperature T and Debye length λ_D as scalings. Using Eqs. (33), (34), and (37), we may write

$$N = \frac{T}{2e^2}\int_0^\infty \rho\, d\rho\, e^{\psi} \equiv \frac{T}{2e^2}F(\gamma) ,$$

$$L = -m\Omega\lambda_D^2\frac{T}{4e^2}\int_0^\infty \rho\, d\rho\, \rho^2 e^{\psi} \equiv -m\Omega\lambda_D^2\frac{T}{4e^2}G(\gamma) ,$$

$$W = \frac{3}{2}NT + \lim_{R\to\infty}\left[\int_0^R 2\pi r\, dr\frac{E^2}{8\pi} - \int_{R_w}^R 2\pi r\, dr\frac{1}{8\pi}\left(\frac{2Ne}{r}\right)^2\right]$$

$$= \frac{T}{4e^2}\left\{3F(\gamma) + \lim_{R\to\infty}\left[\int_0^{R/\lambda_D}\frac{d\rho}{\rho}\left(\int_0^{\rho}\rho'\, d\rho'\, e^{\psi}\right)^2 - F^2(\gamma)\ln(R/\lambda_D)\right] + F^2(\gamma)\ln(R_w/\lambda_D)\right\} \quad (52)$$

$$\equiv \frac{T}{4e^2}[3F(\gamma) + I(\gamma) + F^2(\gamma)\ln(R_w/\lambda_D)] .$$

The functions F, G, and I are displayed in Fig. 2.

The quantities N, L, and W are conserved as the plasma evolves from an arbitrary initial condition to an equilibrium state. Thus, we can uniquely predict the final equilibrium if Eqs. (52) can be inverted; that is, if we can obtain γ, T, and λ_D as functions of N, L, and W. This may be accomplished as follows: From the first and second of Eqs. (52), $\lambda_D^2/R_w^2 = -2LF/m\Omega R_w^2 NG$. Then, the combination of physical parameters given by

$$\frac{W}{e^2N^2} + \frac{1}{2}\ln\left(\frac{-2L}{m\Omega R_w^2 N}\right) = \frac{3}{F(\gamma)} + \frac{I(\gamma)}{F^2(\gamma)} - \frac{1}{2}\ln\left[\frac{F(\gamma)}{G(\gamma)}\right] \equiv D(\gamma) \quad (53)$$

is a function of γ alone. Since $D(\gamma)$ may be seen from Fig. 2 to be one-to-one, γ is uniquely determined by N, L, and W. The first and second of Eqs. (52) then give T and λ_D in terms of $F(\gamma)$ and $G(\gamma)$. Of course, the three parameters γ, T, and λ_D uniquely determine the original parameters n_0, T, and ω, from Eqs. (50) and (48). Knowledge of the total density, angular momentum, and energy for arbitrary initial conditions thus uniquely determines the final equilibrium state. Furthermore, the equilibrium state can be varied in a predictable way by external perturbations which change the total density, angular momentum, or energy.[15]

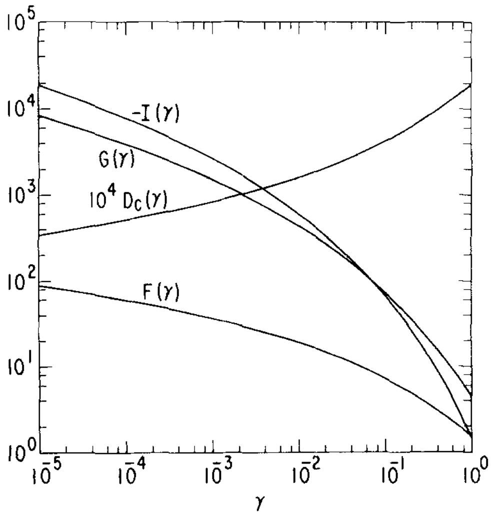

FIG. 2. The functions $F(\gamma)$, $G(\gamma)$, $I(\gamma)$, and $D_c(\gamma)$ vs the equilibrium parameter γ. Here, $D_c(\gamma) \equiv D(\gamma) - (\frac{1}{4} - \frac{1}{2}\ln 2)$.

V. NUMERICAL SOLUTIONS

We now illustrate the ideas of the previous sections by numerical computation of the plasma evolution, from specified initial conditions to the corresponding thermal equilibrium. First, we use the equilibrium parametrization equations of Sec. IV to predict the expected equilibrium, then we numerically integrate the coupled particle transport, energy, and Poisson's equations. We see that $n(r,t)$ and $T(t)$ asymptotically approach the predicted equilibrium density profile and temperature.

We take initial conditions which approximate those which may be obtained experimentally.[3] Consider a cylindrical tube with radius $R_w = 2.5$ cm filled with an electron gas of temperature $T(0) = 1$ eV, with a density profile given by

$$n(r, 0) = 5\times 10^6 \exp(-r^4)\ \mathrm{cm}^{-3} .$$

Then, the number of particles and (lowest order) angular momentum per unit length may be obtained analytically from Eqs. (33) and (34) as $N = 1.39\times 10^7$ cm^{-1} and $L/m\Omega = -3.93\times 10^6$ cm. The energy integral of Eq. (37) must be evaluated numerically, and is $W = 5.20 \times 10^7$ eV-cm^{-1}. For perspective, we note that in this example, the thermal energy ($\frac{3}{2}NT$) is about 2/5 of the total energy W.

The particular thermal equilibrium corresponding to these values of N, $L/m\Omega$, and W may now be obtained from Eqs. (52) and (53). Interestingly, the equilibrium is (to lowest order in ϵ) essentially independent of the magnetic field parameter $m\Omega = eB/c$. Equation (53) gives $D(\gamma) = 0.661$, using $e^2 = 1.44\times 10^{-7}$ eV-cm. This implies $\gamma = 0.282$, $F(\gamma) = 4.06$, $G(\gamma) = 25.7$, and $I(\gamma) = -16.5$, as shown in Fig. 2. The first and second of

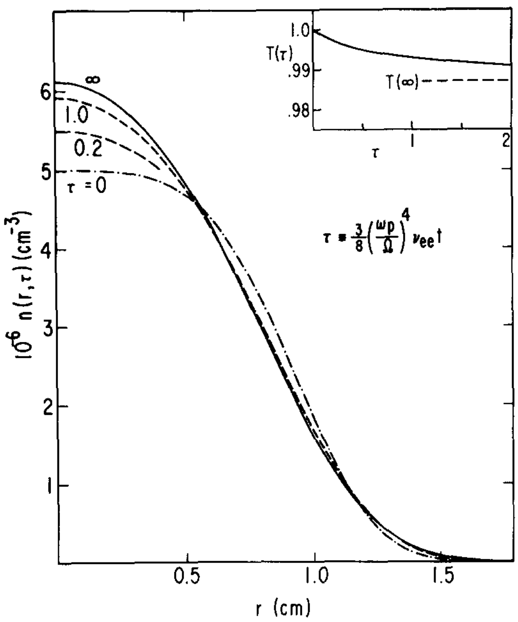

FIG. 3. Density profiles $n(r, \tau)$ as the plasma evolves from an initial condition to the corresponding equilibrium (only part of the $\tau = 0.2$ profile is drawn). Insert shows the temperature $T(\tau)$.

Eqs. (52) then give the equilibrium values $T(\infty) = 0.987$ eV and $\lambda_D = 0.299$ cm. The equilibrium central density is given by Eq. (48) as $n_0 = 6.11 \times 10^6$ cm^{-3}; and the rigid rotor angular velocity ω is given by Eqs. (50) and (42) as $m\omega\Omega = 1.13 \times 10^{-11}$ g-sec^{-2}.

The analysis of Sec. II showed that the particle transport occurs on the time scale $(1/\Omega)(\partial/\partial t) \approx O(\epsilon^4 \delta)$; this is easily seen from Eq. (29) by approximating $\partial/\partial r$ by $1/\lambda_D$. We therefore define a scaled time

$$\tau \equiv \tfrac{3}{8}(\omega_p/\Omega)^4 \nu_{ee} t \ ,$$

so that $\partial/\partial\tau \approx O(1)$. Here, the plasma frequency ω_p and collision frequency ν_{ee} are both evaluated in terms of the initial central density $n(0, 0)$.

The partial differential transport equation (29) is integrated forward in time from $n(r, 0)$, using an iterative implicit scheme for the time step. The electric field is determined at each time step by Poisson's equation (46), and the temperature is computed every few steps from conservation of energy in Eq. (37). At $r = 0$, the various derivatives are evaluated using cylindrical symmetry. The wall is at a radius where $n(r, t) = 0$, to the accuracy of computation.

The computational results are shown in Fig. 3. The central density increases due to inward radial particle fluxes; momentum is conserved by outward fluxes in the low density tail. As these fluxes do net work on the electric field, the temperature decreases slightly. The density profile and temperature are asymptotically approaching their equilibrium values, to the accuracy of computation.

The plasma density in this example is relaxing toward equilibrium on a $\tau = 1$ time scale because most of the transport occurs in regions where $n(r, t) \approx n_0$. However, this would not be the case if the initial plasma had a significant amount of angular momentum or energy in a low density tail. Since the collision frequency is proportional to the density, the relaxation time could be the longer time required for transport at low density.

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APPENDIX

The diffusion of like particles across a magnetic field has been treated in slab geometry by Simon,[5] Longmire and Rosenbluth,[6] and Braginskii.[7] These authors, using differing methods, obtained particle flux results which agreed in form, but differed in numerical coefficients. In this appendix, we reduce our transport results to slab geometry, and obtain results which agree with Ref. 5. We are able to see that the coefficient obtained in Ref. 6 is too large by a factor of 4/3 due to a subtle error in the assumed distribution function.

We first reduce our distribution function results to slab geometry with no electric field and constant temperature. This is accomplished by letting $r \to \infty$, $v_d \to 0$, $\partial T/\partial r \to 0$, and relabeling r by x, and θ by y. We use a prime to denote differentiation with respect to x, that is, $n' = \partial n(x, t)/\partial x$. Equations (8), (10), (14), (18), and (19) then become

$$f_{00} = n(x, t)\left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{mv^2}{2T}\right), \quad f_{10} = -\frac{v_y}{\Omega}\frac{n'}{n} f_{00} \ ,$$

$$f_{20} = -\frac{1}{4\Omega^2}\left[(v_x^2 - v_y^2)\frac{n''}{n} - \left(v_x^2 + v_y^2 - \frac{2T}{m}\right)\left(\frac{n'}{n}\right)^2\right] f_{00} \ , \qquad \text{(A1)}$$

$$f_{21} = -\frac{1}{8\Omega^3}\left(\frac{n'}{n}\right)' \tilde{C}(f_{00}, 2v_x v_y f_{00}) \ .$$

For comparison with Ref. 6, we now express this distribution function, without the collisional term f_{21}, in terms of the coordinate of the guiding center of a particle. A particle with phase space coordinates x, $\mathbf{v}$ will have guiding center $X = x - v_y/\Omega$. The density may then be expressed

$$n(x) = n(X + v_y/\Omega) = n(X) + \frac{v_y}{\Omega} n'(X) + \frac{v_y^2}{2\Omega^2} n''(X) + \cdots \ . \qquad \text{(A2)}$$

To order ϵ^2, the (collisionless) distribution function is then

$$\begin{aligned} f &\equiv f_{00} + f_{10} + f_{20} \\ &= \left\{ n(X)\left(1 - \frac{T}{2m\Omega^2}\frac{n'^2(X)}{n^2(X)}\right) + \frac{(v_x^2 + v_y^2)}{4\Omega^2}\left[\frac{n'^2(X)}{n(X)} - n''(X)\right]\right\} \\ &\quad \times \left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{mv^2}{2T}\right) \\ &\equiv \left[N_a(X) + N_b(X)\frac{m}{2T}(v_x^2 + v_y^2)\right]\left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{mv^2}{2T}\right) . \qquad \text{(A3)} \end{aligned}$$

The full distribution f may be thought of as consisting of two classes of particles, a and b. The class a distribution is separable as a density of guiding centers $N_a(X)$ times a Maxwellian velocity distribution, as assumed in Ref. 6. The class b distribution is not of this form and, although only of order ϵ^2, will be seen to decrease the particle flux by the required factor of 1/4.

We now carry through the flux calculation of Ref. 6, generalized to include the two classes a and b. The flux results from steps ΔX in the guiding center position, due to collisions. The net result of these steps is obtained from the stochastic expression for the flux,

$$F = N_a(X)\langle\Delta X\rangle_{aa} - \frac{1}{2}\frac{\partial}{\partial X}[N_a(X)\langle(\Delta X)^2\rangle_{aa}] + N_a(X)\langle\Delta X\rangle_{ab} - \frac{1}{2}\frac{\partial}{\partial X}[N_a(X)\langle(\Delta X)^2\rangle_{ab}] + N_b(X)\langle\Delta X\rangle_{ba} - \frac{1}{2}\frac{\partial}{\partial X}[N_b(X)\langle(\Delta X)^2\rangle_{ba}] . \quad (A4)$$

Here, for example, $\langle\Delta X\rangle_{ab}$ is the average step of a class a particle due to scattering from all class b particles. To calculate this, we note that a particle whose guiding center is at X will have actual position $x = X + v_y/\Omega$; a scattering particle at this same position will have guiding center coordinate $X_s = X + (v_y - v_{sy})/\Omega$. Further, the step ΔX may be seen to be $\Delta X = -\Delta v_y/\Omega$. The average step is then

$$\langle\Delta X\rangle_{ab} = \left(\frac{m}{2\pi T}\right)^3 \int d^3\mathbf{v}\exp\left(-\frac{mv^2}{2T}\right) \times \int d^3\mathbf{v}_s N_b\left(X + \frac{v_y - v_{sy}}{\Omega}\right)\frac{m}{2T}(v_{sx}^2 + v_{sy}^2) \times\exp\left(-\frac{mv_s^2}{2T}\right)\int d\Omega_s\,\sigma|\mathbf{v} - v_s|(-\Delta v_y/\Omega) = \frac{9}{4}\frac{\bar{v}^2}{\Omega^2}\left(\frac{16\sqrt{\pi}\,e^4}{15m^2\,\bar{v}^3}\ln\Lambda\right)N_b'(X) . \quad (A5)$$

Here, the velocity and angular integrals have been evaluated similarly to those for Eq. (23), as described in detail in Ref. 6. The derivative N_b' arises from the Taylor expansion of $N_b[X + (v_y - v_{sy})/\Omega]$. The final constant in parentheses was identified in Eq. (30) as the collision rate divided by the density, i.e., ν_{ee}/n.

Evaluating the other averages analogously, the guiding center flux of Eq. (A4) may be written

$$\left(\frac{5\bar{v}^2}{2\Omega^2}\frac{\nu_{ee}}{n}\right)^{-1}F = N_a\left(N_a' + \frac{\bar{v}^2}{5\Omega^2}N_a'''\right) - \frac{1}{2}\frac{\partial}{\partial X}\left[N_a\left(N_a + \frac{\bar{v}^2}{5\Omega^2}N_a''\right)\right] + \frac{9}{10}N_aN_b' - \frac{1}{2}\frac{\partial}{\partial X}\left(\frac{8}{10}N_aN_b\right) + \frac{9}{10}N_bN_a' - \frac{1}{2}\frac{\partial}{\partial X}\left(\frac{8}{10}N_bN_a\right) = \frac{\bar{v}^2}{5\Omega^2}(N_aN_a''' - N_a'N_a'') - \frac{\bar{v}^2}{20\Omega^2}(N_aN_a''' - N_a'N_a'') . \quad (A6)$$

Here, we have used $N_aN_b = (\bar{v}^2/2\Omega^2)(N_a'^2 - N_aN_a'')$, from the definition of N_b in Eq. (A3). The lowest order flux terms N_aN_a' are seen to cancel identically due to conservation of momentum. This leaves only the $O(\epsilon^2)$ terms from the Taylor expansion of N_a, and the N_b terms which are inherently $O(\epsilon^2)$. The small "non-Maxwellian" class of guiding centers is thus seen to reduce the flux by 1/4. To lowest order in ϵ, $N_a(X) = n(x)$, so we may write

$$F = \frac{3}{8}\frac{\bar{v}^4}{\Omega^4}\frac{\nu_{ee}}{n}\frac{\partial}{\partial x}\left[n^2\frac{\partial}{\partial x}\left(\frac{1}{n}\frac{\partial n}{\partial x}\right)\right] . \quad (A7)$$

This is the flux we obtained in Eq. (29), reduced to slab geometry.

The same flux was obtained in Ref. 5 from an analysis of the single fluid equations, including off-diagonal terms in the pressure tensor. The pressure tensor for a simple gas in a magnetic field had been derived by Chapman and Cowling[12] from their expansion of the Boltzmann equation. We can easily see that our results give the same pressure tensor and flux.

The steady-state velocity u of the fluid is determined by the momentum balance,

$$\nabla\cdot\mathbf{P} = -(ne/c)\mathbf{u}\times B\hat{z} \quad (A8)$$

where the pressure tensor is

$$\mathbf{P}(x) = m\int d^3\mathbf{v}\,(\mathbf{v} - \mathbf{u})(\mathbf{v} - \mathbf{u})f(x, \mathbf{v}) . \quad (A9)$$

By assumption, only the x derivative in Eq. (A8) is nonzero, so we need the pressure components P_{xx} and P_{xy}; P_{xz} is zero by symmetry. We see from Eqs. (A1) that the lowest order fluid velocity comes from f_{10}, as $u = -(\bar{v}^2/\Omega n)(\partial n/\partial x)\hat{y}$. The largest pressure component is then $P_{xx} = nT$, from f_{00}. The first contribution to P_{xy} is from f_{21}, giving

$$P_{xy} = \frac{3}{8}\frac{\bar{v}^4}{\Omega^3}\frac{\nu_{ee}}{n}mn^2\frac{\partial}{\partial x}\left(\frac{1}{n}\frac{\partial n}{\partial x}\right) . \quad (A10)$$

We note that ν_{ee} is defined[5,12] such that the viscosity in the absence of a magnetic field is $\mu = 2nT/3\nu_{ee}$.

The x component of Eq. (A8) then expresses the consistency between P_{xx} and u_y,

$$\frac{\partial}{\partial x}P_{xx} = T\frac{\partial n}{\partial x} = \frac{neB}{c}\frac{\bar{v}^2}{\Omega n}\frac{\partial n}{\partial x} . \quad (A11)$$

The particle flux nu_x is obtained from the y component of Eq. (A8), as

$$nu_x = -\frac{c}{eB}\frac{\partial}{\partial x}P_{xy} = \frac{3\bar{v}^4}{8\Omega^4}\frac{\nu_{ee}}{n}\frac{\partial}{\partial x}\left[n^2\frac{\partial}{\partial x}\left(\frac{1}{n}\frac{\partial n}{\partial x}\right)\right] . \quad (A12)$$

This is the flux of Ref. 5, since the collision time $\tau = \nu_{ee}^{-1}$.

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