Theory of Electrostatic Fluid Modes in a Cold Spheroidal Non-Neutral Plasma

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The normal modes of a magnetized spheroidally shaped pure ion plasma have recently been measured. Here the theory of these modes is presented. Although one might expect that a numerical solution is required (because the plasma dielectric is anisotropic and the plasma is inhomogeneous), the problem is actually separable in an unusual coordinate system. The result is a simple electrostatic fluid dispersion relation for modes in a cloud of any spheroidal shape.

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In recent experiments[1] a non-neutral plasma is confined for long periods of time in a Penning trap. The plasma is at sufficiently low temperature T and sufficiently high density n_0 so that both the Debye length $\lambda_D = (kT/4\pi q^2 n_0)^{1/2}$ and the interparticle spacing $n_0^{-1/3}$ are much smaller than the size of the plasma (here q is the ion charge). However, the plasma is itself much smaller than the distance of the trap electrodes, so that induced image charges in the electrodes have a negligible effect on the plasma dynamics.

Normal modes have recently been excited and measured in such a plasma cloud.[2] This paper describes a simple analytic theory for these modes. Although the magnetized plasma dielectric is anisotropic (with cylindrical symmetry) and the plasma is also bounded (with a different—spheroidal—symmetry), we show that a separable solution for the partial differential equation governing the mode potential exists, albeit in a rather unusual new coordinate system.

The result is an analytic solution for electrostatic fluid modes in a realistic confined non-neutral plasma of finite size. Previous discussions of normal modes in finite-length plasmas have employed approximations based on periodic boundary conditions applied to an infinite column,[3] or perturbation theory around an approximate finite-length equilibrium.[4] Here exact results are obtained using a realistic thermal equilibrium from which, in the limit of very prolate clouds, one may recover the familiar Trivelpiece-Gould dispersion relation[5] for a cylindrical non-neutral plasma, as well as finite-length corrections to this relation. In the opposite extreme oblate limit, the modes connect onto the dispersion relation of a magnetized slab. In between these limits the modes exhibit characteristics of both relations.

The modes considered here are similar in some respects to those which appear in discussions of the stability of the MacLaurin and Freeman spheroids in the theory of gravitating systems,[6] because the gravitational potential also satisfies a Poisson equation. However, since the equation of state used here is much simpler than that required for gravitational equilibrium (we take the thermal pressure to be identically zero), our normal-mode problem reduces to a separable problem in the theory of electrostatics.

The confinement properties of non-neutral plasmas in Penning traps have been studied extensively. Radial confinement is provided by a strong uniform magnetic field **B**, oriented along the trap axis (taken here to be the z direction). The plasma rotates through this magnetic field, providing a confining $\mathbf{v}\times\mathbf{B}$ force which balances the repulsive radial electric force of the unneutralized plasma. Confinement in the z direction is provided by dc voltages applied to end electrodes. The assumption that the plasma is small implies charges in the electrodes can be neglected and the trap potential is approximately quadratic, of the form $z^2-(x^2+y^2)/2$. Furthermore, the existence of a confined thermal equilibrium state for such plasmas has been demonstrated both theoretically[7] and experimentally.[8,9] In thermal equilibrium, the plasma rotates with a uniform "rigid" rotation frequency ω_r, and is at constant temperature T. If both λ_D and $n_0^{-1/3}$ are much less than the size of the plasma, one may neglect the effects of finite temperature and correlations and the resulting cold-fluid thermal equilibrium is a uniform-density spheroid (ellipsoid of revolution) whose aspect ratio α determines the cloud's rotation frequency ω_r for given trap fields [see Eq. (15) of Ref. 9]. (The aspect ratio α is defined in terms of the cloud's axial length $2b$ and its diameter $2a$ as $\alpha=b/a$.) The rotation frequency in turn is related to the density[7] by $\omega_p^2=2\omega_r \times(\Omega_c-\omega_r)$, where $\Omega_c=qB/Mc$ is the cyclotron frequency, $\omega_p=(4\pi q^2 n_0/M)^{1/2}$ is the plasma frequency, and M is the mass of the charges in the plasma.

In order to treat linear dynamics around this equilibrium, cold-fluid equations are employed in a frame rotating with the plasma. Under the assumption that the perturbed potential ψ has a time dependence of the form $\exp(-i\omega t)$ (in the rotating frame) a differential equation follows from standard manipulations of the continuity, momentum, and Poisson equations:[5]

$$\nabla\cdot\boldsymbol{\varepsilon}\cdot\nabla\psi=0\,, \qquad (1a)$$

where

$$\boldsymbol{\varepsilon}=\begin{pmatrix}\varepsilon_1 & -i\varepsilon_2 & 0\\ i\varepsilon_2 & \varepsilon_1 & 0\\ 0 & 0 & \varepsilon_3\end{pmatrix} \quad (1\text{b})$$

is the plasma dielectric tensor, $\varepsilon_1=1-\omega_p^2/(\omega^2-\Omega_v^2)$, $\varepsilon_2=\Omega_v\omega_p^2/\omega(\omega^2-\Omega_v^2)$, $\varepsilon_3=1-\omega_p^2/\omega^2$, and $\Omega_v=\Omega_c-2\omega_r$ is the "vortex frequency." Equation (1a) is just Maxwell's equation $\nabla\cdot\mathbf{D}=0$ for a medium with linear anisotropic dielectric tensor ε.

Outside the cloud the potential ψ^o satisfies Laplace's equation

$$\nabla^2\psi^o=0\,, \quad (2\text{a})$$

whereas inside the cloud Eq. (1) becomes

$$\varepsilon_1\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi^i+\varepsilon_3\frac{\partial^2}{\partial z^2}\psi^i=0\,. \quad (2\text{b})$$

The solutions inside and outside the cloud are coupled by integration of Eq. (1a) across the plasma vacuum interface S:

$$\psi^i=\psi^o|_S\,, \quad (3\text{a})$$

$$\hat{\mathbf{n}}\cdot\boldsymbol{\varepsilon}\cdot\nabla\psi^i=\hat{\mathbf{n}}\cdot\nabla\psi^o|_S\,, \quad (3\text{b})$$

where $\hat{\mathbf{n}}$ is a unit vector normal to S. Equations (2) and (3), together with the condition that $\psi^o\rightarrow 0$ at large distances, constitute the eigenmode problem for the plasma.

Because of the spheroidal symmetry of the plasma and the cylindrical symmetry of Eq. (2b), this eigenmode problem does not appear to be separable at first glance. However, the problem *is* separable in a rather unusual coordinate system. Cylindrical symmetry implies that the dependence of ψ on the azimuthal angle ϕ can be expressed as $\exp(im\phi)$, where m is any integer. Furthermore, outside the plasma the cloud shape suggests the use of spheroidal coordinates (ξ_1,ξ_2,ϕ), defined by the relations[10]

$$x=[(\xi_1^2-d^2)(1-\xi_2^2)]^{1/2}\cos\phi\,,$$
$$y=[(\xi_1^2-d^2)(1-\xi_2^2)]^{1/2}\sin\phi,\quad z=\xi_1\xi_2\,. \quad (4)$$

Here, d is a free parameter of the coordinate system, chosen as $d=(b^2-a^2)^{1/2}$ in order to make S a surface of constant ξ_1, given by $\xi_1=b$. Surfaces of constant ξ_1 are nested confocal spheroids which become spheres at large distances (the foci are a distance $|d|$ apart). Surfaces of constant ξ_2 are hyperboloids which are everywhere normal to the constant-ξ_1 surfaces. Equation (2a) is separable in these coordinates and the solution compatible with the boundary condition at infinity is

$$\psi^o=AQ_l^m(\xi_1/d)P_l^m(\xi_2)e^{i(m\phi-\omega t)}\,, \quad (5)$$

where P_l^m and Q_l^m are associated Legendre functions, and l is a non-negative integer (larger than or equal to $|m|$) which specifies the number of oscillations in the potential in the ξ_2 direction [$P_l^m(x)$ has $l-|m|$ zeros in the range $-1<x<1$].

However, inside the plasma Eq. (2b) is *not* separable in these spheroidal coordinates. Because the matching conditions, Eqs. (3), mix the inner and outer solutions and would appear to require the use of these coordinates, one might despair of finding a separable solution for this problem. However, one can find a separable solution by applying the following coordinate transformation within the cloud: $\bar{z}=z(\varepsilon_1/\varepsilon_3)^{1/2}$, $\bar{x}=x$, $\bar{y}=y$. In these barred coordinates Eq. (2b) becomes Laplace's equation. Furthermore, since S is taken to another spheroid by this transformation, it is useful to define "scaled" spheroidal coordinates $(\bar{\xi}_1,\bar{\xi}_2,\phi)$ which apply within the cloud:

$$x=[(\bar{\xi}_1^2-\bar{d}^2)(1-\bar{\xi}_2^2)]^{1/2}\cos\phi\,,$$
$$y=[(\bar{\xi}_1^2-\bar{d}^2)(1-\bar{\xi}_2^2)]^{1/2}\sin\phi, \quad (6)$$
$$z(\varepsilon_1/\varepsilon_3)^{1/2}=\bar{\xi}_1\bar{\xi}_2\,.$$

Here, $\bar{d}$ is chosen to be $(\bar{b}^2-a^2)^{1/2}$, where $\bar{b}=b(\varepsilon_1/\varepsilon_3)^{1/2}$, so that the surface of the plasma is also a surface of constant $\bar{\xi}_1$, given by $\bar{\xi}_1=\bar{b}$. Coordinate surfaces are shown in Fig. 1. There are three cases. (1) $\varepsilon_3/\varepsilon_1>0$, $a^2>\varepsilon_3/\varepsilon_1$: Then $\bar{\xi}_1\in\{[-\bar{b},-\bar{d}],[\bar{d},\bar{b}]\}$ and constant-$\bar{\xi}_1$ surfaces are prolate near the cloud center. (2) $\varepsilon_3/\varepsilon_1>0$, $a^2<\varepsilon_3/\varepsilon_1$: Then $\bar{\xi}_1\in[-\bar{b},\bar{b}]$ and constant-$\bar{\xi}_1$ surfaces are oblate. In both cases $\bar{\xi}_2\in[-1,1]$. (3) $\varepsilon_3/\varepsilon_1<0$: $\bar{b}$ is imaginary and the coordinates are double valued. Within the cloud either $\bar{\xi}_1\in[-\bar{b},\bar{b}]$ and $\bar{\xi}_2\in\{[-1,-\bar{b}/\bar{d}],[\bar{b}/\bar{d},1]\}$ or $\bar{\xi}_1\in\{[-\bar{d},-\bar{b}],[\bar{b},\bar{d}]\}$ and $\bar{\xi}_2\in[-\bar{b}/\bar{d},\bar{b}/\bar{d}]$. Now the points $(\bar{\xi}_1,\bar{\xi}_2,\phi)$ and $(\bar{d}\bar{\xi}_2,\bar{\xi}_1/\bar{d},\phi)$ give the same (x,y,z) [see Eq. (4)], so there are two equivalent values of $\bar{\xi}_1$ and $\bar{\xi}_2$ for every point inside the cloud.

While the coordinates are double valued in case 3, the

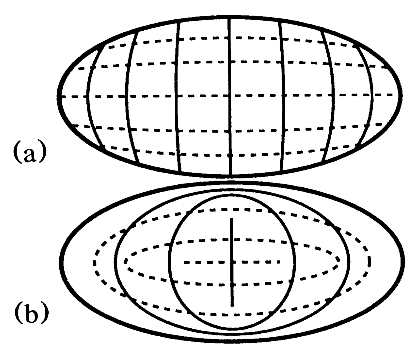

FIG. 1. Qualitative behavior of coordinate surfaces within the cloud. (a) $\varepsilon_3/\varepsilon_1<0$. Solid and dotted lines can be either surfaces of constant $\bar{\xi}_1$ and $\bar{\xi}_2$, respectively, or $\bar{\xi}_2$ and $\bar{\xi}_1$, respectively (see text) (b) Constant-$\bar{\xi}_1$ surfaces for $\varepsilon_3/\varepsilon_1>0$. Dotted lines, $a<(\varepsilon_3/\varepsilon_1)^{1/2}$; solid lines, $a>(\varepsilon_3/\varepsilon_1)^{1/2}$.

solution of Eq. (2b) is a symmetric product of Legendre functions,

$$\psi^i = BP_l^m(\bar{\xi}_1/\bar{d})P_l^m(\bar{\xi}_2)e^{i(m\phi-\omega t)}, \qquad (7)$$

so ψ^i remains single valued. Furthermore, it is not difficult to show that the second matching condition, Eq. (3b), can be expressed in the following simple form:

$$\left(\varepsilon_3\bar{b}\frac{\partial}{\partial\bar{\xi}_1} - i\alpha^2\varepsilon_2\frac{\partial}{\partial\phi}\right)\psi^i\Big|_{\bar{\xi}_1=\bar{b}} = b\frac{\partial}{\partial\xi_1}\psi^o\Big|_{\xi_1=b}. \qquad (8)$$

The appearance of the ξ_1 and $\bar{\xi}_1$ derivatives is not surprising since both derivatives are partials in the directions of $\hat{\mathbf{n}}$. The ϕ derivative arises from the off-diagonal elements of ε.

A simple equation for the mode frequencies follows from substitution of Eqs. (5) and (7) into Eqs. (3a) and (8). Here it is important to note that Eqs. (4) and (6) imply that $\bar{\xi}_2=\xi_2$ on S, so ξ_2 dependences cancel. The resulting equations for A and B are

$$BP_l^m = AQ_l^m, \qquad (9a)$$

$$B[(\bar{b}/\bar{d})\varepsilon_3 P_l^{m\prime} + m\alpha^2\varepsilon_2 P_l^m] = A(b/d)Q_l^{m\prime}. \qquad (9b)$$

Here and in following equations, $P_l^m = P_l^m(\alpha/(\alpha^2-\varepsilon_3/\varepsilon_1)^{1/2})$, $Q_l^m = Q_l^m(\alpha/(\alpha^2-1)^{1/2})$, and primes denote differentiation with respect to the entire argument. The condition for a nontrivial solution to Eqs. (9) leads immediately to the eigenvalue equation

$$\varepsilon_3 + m\alpha\left(\alpha^2 - \frac{\varepsilon_3}{\varepsilon_1}\right)^{1/2}\frac{P_l^m}{P_l^{m\prime}}\varepsilon_2 = \left(\frac{\alpha^2-\varepsilon_3/\varepsilon_1}{\alpha^2-1}\right)^{1/2}\frac{P_l^m Q_l^{m\prime}}{P_l^{m\prime}Q_l^m}. \qquad (10)$$

Equation (10) may be regarded as a finite-length generalization of the familiar Trivelpiece-Gould (TG) dispersion relation for an infinitely long cylindrical nonneutral plasma column, for the case that image charges on the electrodes are unimportant.[10] Indeed, one may obtain this relation from Eq. (10) by taking the cylindrical limit $\alpha\to\infty$, $l\to\infty$, keeping the axial wavelength $k_z = l/\alpha a$ finite. In this limit $\xi_1/d \to 1+\rho^2/2a^2\alpha^2$ (where ρ is the cylindrical radius), $\xi_2 \to z/\alpha a$, $P_l^m \to (-il)^m I_m((\varepsilon_3/\varepsilon_1)^{1/2}k_z a)$, and $Q_l^m \to (il)^m K_m(k_z a)$. Substitution of the latter two limits into Eq. (10) leads to the required relation.

The opposite, magnetized-slab (MS) limit $\alpha\to 0$, $l\to\infty$, $k_\perp = \alpha l/b$ finite, is also of interest. In this limit $\xi_2 \to 1-\rho^2\alpha^2/2b^2$, $\xi_1\to z$, $P_l^m \to l^m\sqrt{2/\pi l}\cos[(l+m)\pi/2 - ik_\perp\bar{b}]$, and $Q_l^m \to (-i)^{l+m+1}l^m\sqrt{\pi/2l}\exp(-k_\perp b)$. The resulting modes induce ripples in the slab of the form

$$J_m(k_\perp\rho)\cos[(l+m)\pi/2 - ik_\perp\bar{z}]\exp(im\phi).$$

In order to understand the behavior of the normal modes as parameters are varied, it is instructive to plot the mode frequencies as a function of l for given m, Ω_v, and α. Such a diagram is shown qualitatively in Fig. 2 for any $m\neq 0$. (For $m=0$ the diagram is symmetric about $\omega=0$. It looks like the top half of Fig. 2 reflected into the bottom half, if one then removes the pair of modes with lowest $|\omega|$.) For finite α the diagram is qualitatively similar to both the TG and MS relations, with frequencies quantized by finite size (i.e., finite length in the case of the former relation, and finite radius in the latter case). If one connects modes which have the same number of radial oscillations (in the $z=0$ plane), one obtains curves similar to the TG relation (the solid curves in Fig. 2). Within a given curve the number of axial oscillations (along $\rho=0$) increases linearly with l, as expected, since $l\propto k_z$ in the cylindrical limit. However, if one instead connects modes with the same number of axial oscillations, one obtains curves similar to the MS relation (the dotted curves in Fig. 2). Now as l increases the number of radial oscillations increases along a curve, since $l\propto k_\perp$ in the MS relation.

These curves may be classified according to the sign of

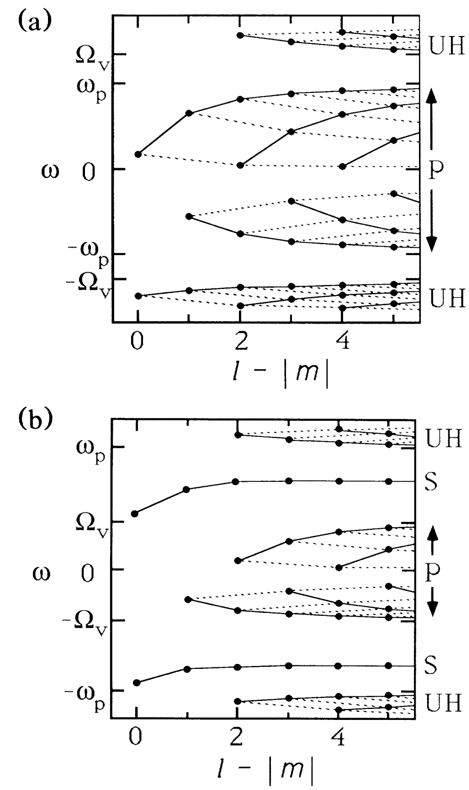

FIG. 2. Qualitative behavior of mode frequences for any $m\neq 0$, $\alpha\sim 1$. $m\Omega_v>0$ assumed. For $m\Omega_v<0$ invert the diagram about the $\omega=0$ axis. Solid and dotted connecting lines are described in text. Modes are labeled P for plasma, UH for upper hybrid, and S for surface plasma. (a) $|\Omega_v|>\omega_p$. (b) $|\Omega_v|<\omega_p$.

$\varepsilon_3/\varepsilon_1$. From Eq. (2b) modes propagate in the plasma when $\varepsilon_3/\varepsilon_1 < 0$, which occurs in the upper hybrid regime

$$\max[\omega_p, |\Omega_v|] < |\omega| < \Omega_{\rm UH} = (\omega_p^2 + \Omega_v^2)^{1/2},$$

and in the range $|\omega| < \min[\omega_p, |\Omega_v|]$, the regime of magnetized plasma oscillations (see Fig. 2). These names refer to mode behavior when $|\Omega_v| > \omega_p$. However, when $\Omega_v \to 0$ (i.e., at the Brillouin limit) the upper hybrid modes become bulk plasma oscillations at $\omega = \omega_p$, and the "magnetized plasma" modes vanish into the $\omega = 0$ resonance.

In addition, evanescent (surface) modes with $\varepsilon_3/\varepsilon_1 > 0$ exist if $|\Omega_v| < \omega_p$, in the range $|\Omega_v| < |\omega| < \omega_p$. These modes are oscillatory in $\bar{\xi}_2$ and evanescent in $\bar{\xi}_1$ [see Fig. 1(b)], and they induce incompressible deformations of the cloud in the limit $\Omega_v \to 0$ (i.e., $\nabla^2 \psi^i = 0$ in this limit). Furthermore, in the limit that $l \to \infty$, asymptotic analysis of the surface modes implies $\omega^2 \to \Omega_{\rm UH}^2/2$, independent of the shape of the cloud.

It is not difficult to show that Eq. (10) can always be written as a polynomial equation in ω, which assists in finding and counting roots. When $m = 0$ the polynomial is of degree l in ω^2, so there are $2l$ modes. When $m \neq 0$ and $l - m$ is even there are $2(l-m)+2$ modes, and if $l - m$ is odd, there are $2(l-m)+1$ modes (see Fig. 2). For example, when $l = |m| \neq 0$ Eq. (10) reduces to a polynomial in ω of degree 2. The equation is, for $m > 0$, $l\alpha(\varepsilon_2 + \varepsilon_1) = \beta$, where $\beta = (\alpha^2 - 1)^{-1/2} Q_l^{l\prime}/Q_l^l$. One root is a diocotron mode, the other is an upper hybrid mode. In the limit $|\Omega_v| \to \infty$ this equation implies that the diocotron mode has frequency $\omega = (1 - \beta/l\alpha)^{-1} \omega_p^2/\Omega_v$ ($m > 0$), giving finite-length corrections to the frequency. It is also instructive to note that when $l = |m|$ one can obtain a simple form for ψ^i in cylindrical coordinates by using Eqs. (6) and (7), together with the fact that $P_l^l(x) \propto (1 - x^2)^{l/2}$. One finds, after some simple algebra, that $\psi^i = C\rho^{|m|} \exp[i(m\phi - \omega t)]$, where C is a constant, so the $l = |m|$ modes produce z-independent displacements of the plasma. Thus, these "$k_z = 0$" diocotron and upper hybrid modes are incompressible, and they induce exactly the same displacements within an ellipsoidal cloud as in an infinitely long cylinder.

In conclusion, a simple analytic form has been found for the normal modes of a constant density magnetized plasma of any spheroidal shape. The theory is directly applicable to recent experiments[2] in which the cloud is near thermal equilibrium, and image charges, temperature, and correlation effects are negligible. The theory should also prove valuable as a starting point in interpreting the results of several other experiments. For example, excitation of normal modes in pure positron[11] and pure antiproton plasmas[12] may be a useful nondestructive diagnostic of properties such as average density, temperature, or plasma shape. Furthermore, plasma modes play an important role in field-error-induced transport[2,13] and transport to thermal equilibrium.[8,14] In these cases this work provides an attractive analytic benchmark for the effects of finite plasma size on the modes.

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