Comment on the stability theorem of Davidson and Lund

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Recently Davidson and Lund[1] have used Arnol'd's method to study nonlinear stability of cold electron plasmas confined by a uniform magnetic field $\mathbf{B}$ and surrounded by a cylindrical conductor. Specifically, they show that an equilibrium in a frame rotating at frequency ω is stable under $\mathbf{E}\times\mathbf{B}$ motion if the density $n(\mathbf{r})$ is functionally related to the streamfunction

$$\psi(\mathbf{r}) = -e\phi(\mathbf{r}) + \omega(eB/2c)r^2, \tag{1}$$

by $n = f(\psi)$ such that $df/d\psi \leqslant 0$.

In the following we use a convexity argument to show that such an equilibrium must be axisymmetric ($\partial n/\partial\theta = 0$). The equilibrium satisfies

$$\nabla^2\psi = -4\pi e^2 f(\psi) + 2\omega(eB/c), \tag{2}$$

with $\psi = \omega(eB/2c)r_\omega^2 = \psi_\omega = \text{const}$ on the cylindrical conductor at $r = r_\omega$. Suppose we have a nonaxisymmetric ($\partial/\partial\theta \neq 0$) solution ψ_1. Then by rotating it through an arbitrary azimuthal angle we obtain another solution ψ_2 that satisfies the same boundary conditions. Except for special choices of rotation angle, ψ_1 and ψ_2 will be distinct. Now

$$\begin{aligned} 0 &< \int d^2\mathbf{r}\,|\nabla(\psi_1 - \psi_2)|^2 \\ &= -\int d^2\mathbf{r}(\psi_1 - \psi_2)\nabla^2(\psi_1 - \psi_2) \\ &= 4\pi e^2 \int d^2\mathbf{r}(\psi_1 - \psi_2)[f(\psi_1) - f(\psi_2)]. \end{aligned} \tag{3}$$

If $df/d\psi \leqslant 0$ there is a contradiction, by the mean value theorem.

To explain this more physically, we can interpret the criterion of Ref. 1 as a statement that the equilibrium minimizes the electrostatic energy in a rotating frame. For an axisymmetric equilibrium with $dn/dr \leqslant 0$, this may be accomplished by choosing a sufficiently large rotation rate so that the well provided by the r^2 term in Eq. (1) overwhelms the ϕ term (ω larger than the $\mathbf{E}\times\mathbf{B}$ rotation rate anywhere in the plasma). The nonaxisymmetric solutions typically of interest, however, are related to diocotron modes that are resonant with or slower than the plasma rotation. In some cases, nonaxisymmetric plasma distributions may be stable because they *maximize* the electrostatic energy in a rotating frame; this situation is analyzed in detail in a forthcoming paper.[2]

An alternative proof that if $df/d\psi \leqslant 0$ only axisymmetric equilibria are possible proceeds as follows. We define an action functional $A[\psi]$ by

$$A[\psi] = \int d^2\mathbf{r}\left(\frac{|\nabla\psi|^2}{2} - 4\pi e^2 \int_{\psi_\omega}^{\psi} d\eta\, f(\eta) + 4\frac{\psi_\omega}{r_\omega^2}(\psi - \psi_\omega)\right). \tag{4}$$

It is straightforward to show that extrema (maxima, minima, or saddle point) of the action functional $A[\psi]$, i.e., $\delta A[\psi] = 0$, correspond to solutions to Poisson's equation (2). Two successive variations of $A[\psi]$ give

$$\delta(\delta A[\psi]) = \int d^2\mathbf{r}\left(|\nabla\delta\psi|^2 - 4\pi e^2 \frac{df(\psi)}{d\psi}(\delta\psi)^2\right). \tag{5}$$

Because the first term in the integrand of Eq. (5) is nonnegative, it follows trivially that $df(\psi)/d\psi \leqslant 0$ is a sufficient condition for $\delta(\delta A[\psi]) > 0$. However, if $\delta(\delta A[\psi]) > 0$ at all possible extrema, it immediately follows that there can only be one extremum (a global minimum) to $A[\psi]$.

Solutions to Eq. (2) can be parametrized by the number of constants necessary to fix the function $f(\psi)$ and the constant value of the streamfunction $\psi = \psi_\omega$ on the cylindrical conductor at $r = r_\omega$. For $df/d\psi \leqslant 0$, it can be shown that if it is assumed that a set of parameters leads to a nonaxisymmetric solution, then there is necessarily at least one axisymmetric solution for the same set of parameters. This situation would correspond to there being at least two extrema to the action functional $A[\psi]$ for these parameters. However, for $df/d\psi \leqslant 0$ there is only one extremum to $A[\psi]$, which contradicts the assumed existence of a nonaxisymmetric solution for $df/d\psi \leqslant 0$. Therefore, $df(\psi)/d\psi \leqslant 0$ is a sufficient condition for any solution to Eq. (2) satisfying $\psi = \psi_\omega = \text{const.}$ at $r = r_\omega$ to be axisymmetric ($\partial/\partial\theta = 0$). It follows that a necessary condition for the existence of rotating equilibria with coherent structure ($\partial/\partial\theta \neq 0$) is $df(\psi)/d\psi \geqslant 0$ or that $df(\psi)/d\psi$ change sign within the plasma region.

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[1] R. C. Davidson and S. M. Lund, Phys. Fluids B **3**, 2540 (1991).

[2] T. M. O'Neil and R. A. Smith, submitted to Phys. Fluids B (1992).