New Theory of Transport Due to Like-Particle Collisions

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Cross-magnetic-field transport due to like-particle collisions is discussed for the parameter regime $\lambda_D \gg r_L$, where λ_D is the Debye length and r_L is the characteristic Larmor radius of the colliding particles. A new theory based on collisionally produced $\mathbf{E}\times\mathbf{B}$ drifts predicts a particle flux which exceeds the flux predicted previously, by the factor $(\lambda_D/r_L)^2 \gg 1$.

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This paper presents a new theory of cross-magnetic-field transport due to like-particle collisions. The new theory invokes a transport mechanism which is quite different from that considered in the traditional theory of like-particle transport.[1-4] The ratio of the particle flux predicted by the new theory to that predicted by the traditional theory is $(\lambda_D/r_L)^2$, where λ_D is the Debye length and r_L is the characteristic Larmor results of the colliding particles; so the new theory supersedes the traditional theory in the parameter regime $\lambda_D \gg r_L$.

This is the typical operating regime for experiments with magnetically confined pure electron plasmas,[4-6] and there is a current effort to measure electron-electron transport in such a plasma.[7] The theory presented here is motivated by these experiments, and the analysis is carried out with a pure electron plasma in mind.

To understand the transport mechanism invoked by the new theory, consider two electrons which interact while moving along field lines which are separated by a distance ρ, where $\lambda_D \geqslant \rho \gg r_L$. One may think of ρ as a kind of impact parameter. The first inequality implies that the interaction is not shielded out, and the second implies that the effect of the interaction on the electron motion may be described by guiding-center drift theory. The guiding center for each electron experiences an $\mathbf{E}\times\mathbf{B}$ drift, and the time integral of the drift velocity over the duration of the interaction yields a step in the guiding-center position. Such steps are the elementary steps of the transport process. The process is similar to the two-dimensional transport of charged rods,[8] except that here the motion of the electrons along the magnetic field lines plays an important role in the dynamics.

Previous discussions of transport due to like-particle collisions, as opposed to like-rod collisions, invoke a quite different mechanism.[1-4] In these discussions, a step in the position of an electron guiding center arises as a result of the collisional scattering of the electron velocity vector. The discussions are based on solutions of the Boltzmann equation (or Lenard-Balescu equation) for a magnetized plasma, and the effect of velocity scattering is included through the collision operator (but the effect of collisionally produced $\mathbf{E}\times\mathbf{B}$ drifts is not included). For the parameter regime $r_L \ll \lambda_D$, velocity scattering is associated primarily with small–impact-parameter collisions (i.e., $\rho \leqslant r_L$), and, in the collision operator, the integral over impact parameters is cut off at $\rho \sim r_L$.[4] Thus, the traditional theory of like-particle transport focuses attention on small–impact-parameter collisions, whereas the new theory focuses attention on large–impact-parameter collisions (i.e., $\rho \sim \lambda_D \gg r_L$).

It is easy to understand why large–impact-parameter collisions are more effective than small–impact-parameter collisions in producing like-particle transport. From conservation of momentum, one can see that the guiding centers for two like particles which are involved in a collision make equal and opposite steps. This is true whether the steps are due to velocity scattering or to $\mathbf{E}\times\mathbf{B}$ drifts. If the two guiding centers are at nearly the same position, the equal and opposite steps contribute nearly canceling contributions to the particle flux. On the other hand, for well separated guiding centers, the equal and opposite steps each contribute locally to the flux. We will see that the net flux is proportional to the square of the separation between the guiding centers. For steps due to velocity scattering, the guiding centers can be separated by no more than r_L, whereas for steps due to $\mathbf{E}\times\mathbf{B}$ drifts, the guiding centers can be separated by as much as λ_D. This is the reason that the ratio of the flux predicted by the new theory to that predicted by the traditional theory is $(\lambda_D/r_L)^2 \gg 1$.

As was mentioned, the effect of collisionally produced $\mathbf{E}\times\mathbf{B}$ drifts is not included in the Boltzmann equation. To include this effect, one must start further back in the analysis, that is, with the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy,[9,10] and that is the approach which we follow.

For simplicity, we consider a slab geometry, characterized by the uniform magnetic field $\hat{\mathbf{z}}B$, the mean self-consistent electric field $\hat{\mathbf{x}}E(x)$, and the mean density $n(x)$. The plasma is assumed to be homogeneous in the y and z directions. The density gradient and the electric field together with the electron-electron interactions result in an electron flux $\Gamma_x(x)$, which we will calculate.

Since the main contribution to the flux comes from collisions for which guiding-center drift theory is well satisfied (i.e., $\rho \sim \lambda_D >> r_L$), we use a guiding-center model to describe the plasma. The state of an electron is specified by $(\mathbf{r}, v)$, where $\mathbf{r} = (x,y,z)$ is the guiding-center position and v is the velocity along a field line. The electrons stream and accelerate along the field lines and undergo $\mathbf{E}\times\mathbf{B}$ drift across them. Because of the ordering $\lambda_D >> r_L$, the polarization drift which occurs during an interaction is negligible compared to the associated $\mathbf{E}\times\mathbf{B}$ drift.

From the one-electron equation of the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy for the guiding-center model (or simply from inspection), one can see that the electron flux in the x direction is given by the expression

$$\Gamma_x(x_1) = -(N/V)^2 \int dv_1 \int dv_2 \int d^3r_2 (c/B)\partial\psi(\mathbf{r}_1, \mathbf{r}_2)/\partial y_1\, f_2(\mathbf{r}_1, v_1, \mathbf{r}_2, v_2, t), \tag{1}$$

where $\psi(\mathbf{r}_1, \mathbf{r}_2)$ is the interaction potential between electrons 1 and 2, $f_2(\mathbf{r}_1, v_1, \mathbf{r}_2, v_2, t)$ is the two-electron distribution function, N is the total number of electrons, and V is the total volume. The two-electron function is normalized in the usual manner (i.e., $V^2 = \int d^3r_1 \int dv_1 \int d^3r_2 \int dv_2 f_2$).

For simplicity, we treat the shielding in an *ad hoc* manner, and write $\psi(\mathbf{r}_1, \mathbf{r}_2)$ as the Debye-shielded Coulomb interaction

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = -(e/|\mathbf{r}_1 - \mathbf{r}_2|)\exp[-|\mathbf{r}_1 - \mathbf{r}_2|/\lambda_D], \tag{2}$$

where $1/\lambda_D^2 = 4\pi e^2 n[\frac{1}{2}(x_1 + x_2)]/T$. The two-electron function then evolves according to the equation

$$\left\{\frac{\partial}{\partial t} + (v_2 - v_1)\frac{\partial}{\partial z_2} + \frac{c}{B}[E(x_1) + E(x_2)]\frac{\partial}{\partial y_2} + \frac{e}{m}\frac{\partial\psi}{\partial z_2}\left(\frac{\partial}{\partial v_2} - \frac{\partial}{\partial v_1}\right) + \frac{c}{B}\frac{\partial\psi}{\partial y_2}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) + \frac{c}{B}\left(\frac{\partial\psi}{\partial x_2} - \frac{\partial\psi}{\partial x_1}\right)\frac{\partial}{\partial y_2}\right\} f_2 = 0, \tag{3}$$

where use has been made of the assumed homogeneity in y and z.

Of course, an *ad hoc* treatment of shielding is not rigorously correct. A proper treatment is complicated mathematically by the fact that the shielding takes place in an inhomogeneous plasma. The purpose of this Letter is not to calculate the precise value of the flux, but rather to illustrate simply that a new transport mechanism yields a flux which is much larger than that obtained previously.

Equation (3) can be solved by use of a perturbation expansion in the interaction strength ψ. To this end, we set $f_2 = f_2^{(0)} + f_2^{(1)}$ and $L = L^{(0)} + L^{(1)}$, where L is the operator which acts on f_2. The superscript (0) refers to quantities which are zero order in ψ and the superscript (1) to quantities which are first order in ψ. Thus, $L^{(0)}$ consists of the first three terms in the bracket of Eq. (3) and $L^{(1)}$ of the last three terms.

In zero order, Eq. (3) reduces to the form $L^{(0)} f_2^{(0)} = 0$, which has the solution $f_2^{(0)} = f_2^{(0)}(x_1, v_1, x_2, v_2)$. The solution is further constrained by the observation that in zero order there can be no electron-electron correlations; the two-electron distribution must be related to the one-electron distribution through the equation $f_2^{(0)}(1,2) = f_1^{(0)}(1) f_1^{(0)}(2)$. Taking the one-electron distribution to be a Maxwellian characterized by density $n(x)$ and temperature T yields the solution

$$f_2^{(0)} = \left(\frac{V}{N}\right)^2 \frac{n(x_1)n(x_2)}{(2\pi T/m)} \exp\left[-\frac{1}{T}\left(\frac{mv_1^2}{2} + \frac{mv_2^2}{2}\right)\right]. \tag{4}$$

Of course, the zero-order $\mathbf{E}\times\mathbf{B}$ flow in the y direction does not appear explicitly in the distribution as a velocity dependence, since the distribution refers to guiding centers. Only a flow in the z direction would appear as a velocity dependence.

It is interesting to note that the Maxwellian character of the one-electron distribution is forced by collisions which are not directly included in the guiding-center model. This model focuses attention on the large–impact-parameter collisions (i.e., $\rho \sim \lambda_D >> r_L$), which yield the dominant contribution to the flux. The scattering of electron velocity vectors, which is described by the collision operator, is due primarily to collisions characterized by small impact parameter (i.e., $\rho \lesssim r_L$). These collisions are not directly included in the guiding-center model but do have the indirect influence of forcing the one-electron distribution to be Maxwellian.

In first order, Eq. (3) reduces to $L^{(0)} f_2^{(1)} + L^{(1)} f_2^{(0)} = 0$, which when written out takes the form

$$\frac{df_2^{(1)}}{dt} = \frac{e}{T}(v_2 - v_1)\frac{\partial\psi}{\partial z_2} f_2^{(0)} + \frac{c}{B}\frac{\partial\psi}{\partial y_2}\left[\frac{1}{n(x_2)}\frac{dn}{dx_2} - \frac{1}{n(x_1)}\frac{dn}{dx_1}\right] f_2^{(0)}, \tag{5}$$

where $d/dt = L^{(0)}$ is the total time derivative along the unperturbed orbit. By use of the relation

$$\frac{d\psi}{dt} = (v_2 - v_1)\frac{\partial\psi}{\partial z_2} - \frac{c}{B}[E(x_2) - E(x_1)]\frac{\partial\psi}{\partial y_2}, \tag{6}$$

the solution to Eq. (5) can be written as

$$f_2^{(1)} = \frac{e\psi}{T}f_2^{(0)} + \left\{\left[\frac{1}{n(x_2)}\frac{dn}{dx_2} - \frac{1}{n(x_1)}\frac{dn}{dx_1}\right] + \frac{e}{T}[E(x_2) - E(x_1)]\right\}f_2^{(0)}\int_{-\infty}^{t} dt' \frac{c}{B}\left(\frac{\partial\psi}{\partial y_2}\right)', \tag{7}$$

where the time integral is along the unperturbed orbit.

By combination of Eqs. (1), (4), and (7), the flux can be written as

$$\Gamma_x(x_1) = \int d^3r_2\left\{\left[\frac{1}{n(x_2)}\frac{dn}{dx_2} - \frac{1}{n(x_1)}\frac{dn}{dx_1}\right] + \frac{e}{T}[E(x_2) - E(x_1)]\right\}n(x_1)n(x_2)h(\mathbf{r}_2 - \mathbf{r}_1, (x_1+x_2)/2), \tag{8}$$

where

$$h(\mathbf{r}_2 - \mathbf{r}_1, (x_1+x_2)/2) = \int dv_1 \int dv_2 \frac{\exp[-T^{-1}(\frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2)]}{(2\pi T/m)}\int_{-\infty}^{t} dt' \left(\frac{c}{B}\right)^2\left(\frac{\partial\psi}{\partial y_2}\right)\left(\frac{\partial\psi}{\partial y_2}\right)'. \tag{9}$$

Because of Debye shielding, the $\mathbf{r}_2$ integral in Eq. (8) receives significant contributions only for x_2 near x_1 (i.e., for $|x_2 - x_1| \lesssim \lambda_D$). We assume that $n(x_2)$ and $E(x_2)$ vary on a length scale which is large compared to λ_D, and we make Taylor expansions of $n(x_2)$ and $E(x_2)$ about $x_2 = x_1$. The variation of $h(\mathbf{r}_2 - \mathbf{r}_1, (x_1+x_2)/2)$ through its second argument is on the same scale as that for $n(x_2)$ and $E(x_2)$; so $h(\mathbf{r}_2 - \mathbf{r}_1, (x_1+x_2)/2)$ also can be Taylor expanded about $x_2 = x_1$. Of course, the dependence of $h(\mathbf{r}_2 - \mathbf{r}_1, (x_1+x_2)/2)$ on its first argument cannot be Taylor expanded; it is the peaked nature of this dependence which justifies the other expansions. Carrying out the expansions and substituting into Eq. (8) yields the result

$$\Gamma_x(x_1) = \frac{d}{dx_1}n^2(x_1)K(x_1)\frac{d}{dx_1}\left[\frac{1}{n(x_1)}\frac{dn}{dx_1} + \frac{e}{T}E(x_1)\right], \tag{10}$$

where $K(x_1) = \int d^3r\,(x^2/2)h(\mathbf{r},x_1)$ is the transport coefficient and $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ is the relative position vector. This expression for the flux differs from that obtained previously only in the value of the transport coefficient $K(x)$, which we now proceed to evaluate.

The unperturbed orbit for the relative position vector is specified by

$$z' = z + v(t' - t),$$
$$y' = y + (c/B)[E(x_2) - E(x_1)](t' - t),$$

and $x' = x$. Here, the relative velocity $v = v_2 - v_1$ and x_1 and x_2 are independent of t'. By use of the relation

$$|E(x_2) - E(x_1)| \lesssim |dE/dx_2|\lambda_D = 4\pi en\lambda_D,$$

one can see that the inequality $\lambda_D >> r_L$ implies the inequality $\bar{v} >> (c/B)|E(x_2) - E(x_1)|$. In other words, for most collisions, the relative velocity parallel to the field is much larger than the relative velocity across the field. As a first approximation, we neglect the relative cross-field motion which occurs during an interaction and set $y' \simeq y$.

By use of the orbit $z' = z + v(t' - t)$, $y' = y$, and $x' = x$ together with a Fourier representation of the interaction potential

$$\psi = \int \frac{d^3k}{(2\pi)^3}\frac{(-4\pi e)}{k^2 + 1/\lambda_D^2(x_1)}\exp(i\mathbf{k}\cdot\mathbf{r}), \tag{11}$$

one can show that

$$K(x_1) = \frac{\pi e^2c^2\lambda_D^2(x_1)}{6B^2}\int \frac{dv}{|v|}\frac{\exp(-mv^2/4T)}{(4\pi T/m)^{1/2}}. \tag{12}$$

Here, the integral over the relative velocity arises in the following way. The integral $\int dv_1 \int dv_2$ is replaced by the integral $\int dv \int dV$, where $V = (v_1 + v_2)/2$ is the center-of-mass velocity, and then the integral over the center-of-mass velocity is carried out.

The integral over the relative velocity is logarithmically divergent at $v = 0$. Physically, this corresponds to the fact that two electrons with small relative velocity interact for a long time and experience large $\mathbf{E}\times\mathbf{B}$ steps due to the interaction. To remove the divergence we must take into account physical effects which limit the time of the interaction, or, equivalently, cut off the velocity integral at some small but finite value of $|v|$ (i.e., $\min|v| = \Delta v$).

One such effect is small-angle scattering. The small–impact-parameter collisions, which are not directly included in the guiding-center model, produce a diffusive spreading of $v = v_2 - v_1$. During the time τ, the amount of spreading is $(\Delta v)^2 = \nu\bar{v}^2\tau$, where ν is the collision frequency and $\bar{v}^2 = T/m$. This velocity spread can separate electrons 1 and 2 by a Debye length during the time τ provided that $(\Delta v)\tau = \lambda_D$. Eliminating τ between the two relations yields the result $(\Delta v/\bar{v}) = (\nu/\omega_p)^{1/3}$, where ω_p is the plasma frequency.

A competing effect is associated with the relative cross-field motion of the two electrons. Two electrons for which $|x_1 - x_2| \simeq \lambda_D$ have a relative y velocity of $(c/B)|dE/dx|\lambda_D = (c\lambda_D 4\pi en)/B$. The time τ for this relative velocity to produce a separation $|y_1 - y_2| \simeq \lambda_D$ is given by $\tau(4\pi enc)/B = 1$. Relating this time to a relative parallel velocity through $(\Delta v)\tau = \lambda_D$ yields the result $(\Delta v/\bar{v}) = r_L/\lambda_D$. Of course, the cutoff for the velocity integral in Eq. (12) is determined by the effect which yields the largest value of $(\Delta v/\bar{v})$.

Introducing the cutoff and carrying out the velocity integral yields the result

$$K(x_1) = \frac{\sqrt{\pi} e^2 c^2 \lambda_D^2(x_1)}{6B^2 \bar{v}} \ln\left|\frac{\bar{v}}{\Delta v}\right|. \tag{13}$$

This coefficient should be compared to the coefficient obtained previously, that is, to $(\frac{3}{8})(\nu/n) r_L^4$, where $\nu = (16\sqrt{\pi} e^4 n/15 m^2 \bar{v}^3)\ln(r_L/b)$ is the collision frequency and $b = e^2/m\bar{v}^2$ is the distance of closest approach. The ratio of the new coefficient to the previous coefficient is given by

$$(\tfrac{5}{12})[\ln(\bar{v}/\Delta v)/\ln(r_L/b)](\lambda_D/r_L)^2.$$

In the parameter regime $\lambda_D \gg r_L$, the new coefficient is much larger than the previous coefficient, and the new coefficient scales with magnetic field strength as $1/B^2$ rather than $1/B^4$. Such dramatic differences should be observable experimentally.

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